The open XXZ and associated models at q root of unity

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Abstract

The generalized open XXZ model at q root of unity is considered. We review how associated models, such as the q harmonic oscillator, and the lattice sine-Gordon and Liouville models are obtained. Explicit expressions of the local Hamiltonian of the spin $\frac{1}{2}$ XXZ spin chain coupled to dynamical degrees of freedom at the one end of the chain are provided. Furthermore, the boundary non-local charges are given for the lattice sine Gordon model and the q harmonic oscillator with open boundaries. We then identify the spectrum and the corresponding Bethe states, of the XXZ and the q harmonic oscillator in the cyclic representation with special non diagonal boundary conditions. Moreover, the spectrum and Bethe states of the lattice versions of the sine-Gordon and Liouville models with open diagonal boundaries is examined. The role of the conserved quantities (boundary non-local charges) in the derivation of the spectrum is also discussed.

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1   Introduction

The ultimate goal when studying a physical system is the derivation of the corresponding observables. In the framework of integrable spin chains the first step towards such an aim is the diagonalization of the corresponding transfer matrix by means of the quantum inverse scattering method, introduced by the St. Petersburg school [1, 2, 3]. The diagonalization process relies primarily on the existence of certain exchange relations emerging from the specific algebra that rules the model, defined by either the Yang-Baxter [3, 4, 5] or the reflection equation [6, 7]. From the physical viewpoint Yang–Baxter and reflection equations describe the factorization of multiparticle scattering in the whole and half line respectively, a unique feature displayed by 2D integrable systems. Due to analyticity requirements imposed upon the spectrum certain constraints, known as Bethe ansatz equations [2, 8] arise, whose exact form depends explicitly on the choice of representation. The significance of Bethe ansatz equations rests on the fact that their solutions yield all the physically relevant quantities such as exact S matrices [2, 9, 10], thermodynamic properties [11, 12, 13] and correlation functions [3, 14]. In addition it was recently realized [15] that in a particular limit string theories may be mapped to $\mathcal{N} = 4$ super Yang-Mills gauge theories, which in turn can be associated to special examples of integrable spin chains. Thus one may implement the powerful Bethe ansatz techniques [2, 8], in order to derive non-perturbatively the physically relevant quantities. This remarkable realization put integrable spin chains and Bethe ansatz into the fore of recent advances in string theory.

The main aim of the present article is the investigation of the spectrum of a generalized XXZ model, and associated models, with special open boundary conditions that preserve integrability. Historically the spin $\frac{1}{2}$ XXZ chain with diagonal boundaries was investigated in [7, 16, 17], whereas the corresponding spin chain with non diagonal boundaries was just recently solved for q root of unity [18], and for generic values of q in [19]. The spectrum and Bethe ansatz for the spin s XXZ model with non-diagonal boundaries were also derived in [20, 21]. In [22] the spectral equivalence between the Hamiltonians of the XXZ model with a non diagonal boundary and a novel open model (the asymmetric twin chain) with an obvious reference state is shown via the representation theory of boundary Temperley–Lieb algebras. Furthermore, the spectral equivalence of XXZ type Hamiltonians with diagonal and non-diagonal boundaries was investigated in [23], again in the context of boundary Temperley-Lieb algebras. Equivalence of the spectra of spin chains with different boundaries was also established in [24] by introducing appropriate defects. In the present study we shall be mainly interested in the case where the parameter q is a root of unity, and we shall restrict our attention in finding the spectrum of the general XXZ model and the q harmonic oscillator with non diagonal boundaries for the so called cyclic representation [25], which has not been treated so far. It will be instructive to provide a special example of the Hamiltonian of the spin $\frac{1}{2}$ XXZ model with special integrable conditions, i.e.

$$\mathcal{H} = -\frac{1}{4} \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh(i\mu) \ \sigma_i^z \sigma_{i+1}^z \right) - \frac{N}{4} \ \cosh(i\mu) + \mathcal{M}_N + c_1 \ \sigma_1^z + c_2 \ \sigma_1^+ + c_3 \ \sigma_1^- \ (1.1)$$

$\sigma^{x,y,z}$, $(\sigma^{\pm})$ are the 2×2 Pauli matrices. The constants c_i are dictated by integrability, and $\mathcal{M}$ is a 2×2 matrix with entries being either scalars (c-number K matrix applied at the right boundary) or elements of a particular algebra, e.g. $\mathcal{U}_q(sl_2)$, the q harmonic oscillator algebra etc., that is a dynamical boundary coupled to the right end of the spin chain, that is a typical quantum impurity problem. In general such

type of Hamiltonians turn out to be of particular significance in condensed matter physics, describing for instance quantum impurities (see [7, 13, 20, 24], [26]–[29] and references therein) or being related to the Azbel-Hofstadter Hamiltonian [26, 30, 31, 32]. Note that we shall also examine the spectrum and Bethe ansatz equations of the open lattice sine Gordon and Liouville models, but only with diagonal boundaries.

Let us outline the content of each section of the present article. Sections 2 and 3 serve basically as a general review. More precisely, in section 2 we introduce the generalized XXZ model, and we also review how associated models such as the lattice sine-Gordon, Liouville models and the q harmonic oscillator arise via certain limiting processes. The underlying algebraic description is also provided and useful realizations are presented. In fact, one of the the main objectives of this article is to offer a unifying framework for the study of a whole class of open integrable lattice models emerging from the generalized XXZ model. However to identify the spectrum of the relevant models one has to fix a representation and this is done in sections 6 and 7. In section 3 section well known c-number K matrices, solutions of the reflection equation are reviewed, and the algebraic transfer matrix of an open spin chain is also recalled.

In section 4 a special example of a XXZ Hamiltonian (1.1), with spin $\frac{1}{2}$ in the bulk coupled to some dynamical system at the one end is considered. More precisely, explicit novel expressions of the dynamical boundary term $\mathcal{M}$ of the Hamiltonian (1.1) are provided. Note that these expressions are generic (algebraic) at this stage, i.e. they are independent of the choice of representation for both the $\mathcal{U}_q(sl_2)$ and the q oscillator algebras, which are our primary interest in the present study. In fact such a Hamiltonian may be regarded as a special example of the more general case described in section 3. To deal with the corresponding spectrum naturally one has to choose a particular representation associated to the boundary. This is done however in section 6, where we restrict our attention to the cyclic representation for both XXZ and the q harmonic oscillator. In section 5 the boundary non local charges for the generic XXZ chain are reviewed, and novel expressions of the non-local charges for the lattice sine-Gordon model and q harmonic oscillator are derived. Their significance and their relation to the spectrum becomes clear in section 6.

In section 6 the spectrum of the XXZ model and the q harmonic oscillator in the cyclic representation [25], for special non-diagonal boundaries is investigated. Generalizing the formulation of [19, 33] we are able to identify a reference state, by means of suitable local gauge transformations known also as Darboux matrices [34], upon which eigenstates are built. The derivation of the reference states, which is a new result, is achieved by solving sets of recursion relations. Within the algebraic Bethe ansatz framework novel expressions of the spectrum and Bethe ansatz equations of the relevant models are found. In this context the Hamiltonian presented in section 4, with the boundary terms associated to the cyclic representation, (see also (1.1)) is also treated as a special case. Another intriguing new result is the connection between the spectrum of the open transfer matrix and the spectrum of the conserved boundary non local charges, which is presented in section 6.3. Although this is a generic result, which hold for any representation we provide particular examples for the XXZ model and the q harmonic oscillator in the cyclic representation, given that these models are our primary concern. In Appendix C we present such a conserved quantity in the spin s representation (locally) as a tridiagonal (Jacobi) matrix [35], and in the cyclic representation of $\mathcal{U}_q(sl_2)$, and we attempt its diagonalization. In section

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7 we give a flavor on the lattice sine-Gordon and Liouville theories with diagonal boundary conditions. This is the first time to our knowledge that these models with open boundary conditions are considered. Finally, in the conclusion section we summarize the results of the present article.

2 The XXZ and associated models

In this section the XXZ model is introduced, and various closely related models, which arise from it naturally are reviewed. The XXZ spin $\frac{1}{2}$ R matrix, associated to the fundamental representation of $\mathcal{U}_q(sl_2)$, acting on $(\mathbb{C}^2)^{\otimes 2}$. In general, the R matrix is a quantity proportional to the physical S matrix, acts on $\mathbb{V}^{\otimes 2}$, and is a solution of the Yang-Baxter equation [3, 4]

$$R_{12}(\lambda_1 - \lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2), \tag{2.1}$$

which acts on $\mathbb{V}^{\otimes 3}$, and as customary $R_{12} = R \otimes \mathbb{I}$, $R_{23} = \mathbb{I} \otimes R$. The spin $\frac{1}{2}$ XXZ R matrix in particular is given by

$$R(\lambda) = \begin{pmatrix} \sinh\mu(\lambda + \frac{i}{2} + \frac{i\sigma^z}{2}) & \sigma^- \sinh(i\mu) \\ \sigma^+ \sinh(i\mu) & \sinh\mu(\lambda + \frac{i}{2} - \frac{i\sigma^z}{2}) \end{pmatrix}. \tag{2.2}$$

The R matrix is written in the so called principle gradation, it can be however expressed in the homogeneous gradation by means of a simple gauge transformation

$$R_{12}^{(h)}(\lambda) = \mathcal{V}_1(-\lambda)\ R_{12}^{(p)}(\lambda)\ \mathcal{V}_1(\lambda), \qquad \mathcal{V}(\lambda) = diag(1,\ e^{\mu\lambda}). \tag{2.3}$$

One may now derive a more general object $L(\lambda) \in \text{End}(\mathbb{C}^2) \otimes \mathcal{U}_q(\widehat{sl_2})$, where the quantum algebra $\mathcal{U}_q(\widehat{sl_2})$ is defined by the fundamental algebraic relation (see also Appendix A)

$$R_{ab}(\lambda_1 - \lambda_2)\ L_{an}(\lambda_1)\ L_{bn}(\lambda_2) = L_{bn}(\lambda_2)\ L_{an}(\lambda_1)\ R_{ab}(\lambda_1 - \lambda_2). \tag{2.4}$$

A simple solution of the latter equation, which we shall use hereafter, is (index free notation)

$$L(\lambda) = \begin{pmatrix} e^{\mu\lambda}\text{A} - e^{-\mu\lambda}\text{D} & (q - q^{-1})\text{B} \\ (q - q^{-1})\text{C} & e^{\mu\lambda}\text{D} - e^{-\mu\lambda}\text{A} \end{pmatrix} \tag{2.5}$$

A, B, C, D satisfy the defining relations of $\mathcal{U}_q(sl_2)$ $(q = e^{i\mu})$, namely

$$\text{A D} = \text{D A} = \mathbb{I}, \quad \text{A C} = q\text{C A}, \quad \text{A B} = q^{-1}\text{B A}, \quad \Big[\text{C, B}\Big] = \frac{\text{A}^2 - \text{D}^2}{q - q^{-1}}. \tag{2.6}$$

A well known representation of $\mathcal{U}_q(sl_2)$ is the spin s representation, which is n = 2s + 1 dimensional, and may be expressed in terms of n × n matrices as

$$A = \sum_{k=1}^{n} q^{\alpha_k}\ e_{kk}, \qquad C = \sum_{k=1}^{n-1} \tilde{C}_k\ e_{k\ k+1}, \qquad B = \sum_{k=1}^{n-1} \tilde{C}_k\ e_{k+1\ k} \tag{2.7}$$

where we define the matrix elements $(e_{ij})_{kl} = \delta_{ik}\ \delta_{jl}$ and

$$\alpha_k = \frac{n+1}{2} - k, \qquad \tilde{C}_k = \sqrt{[k]_q[\text{n} - k]_q}, \qquad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{2.8}$$

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The generators A, B, C, D may be also expressed in terms of the Heisenberg-Weyl group generators $\mathbb{X}$, $\mathbb{Y}$

$$\mathbb{X} \, \mathbb{Y} = q \, \mathbb{Y} \, \mathbb{X} \tag{2.9}$$

as

$$A = D^{-1} = \mathbb{X}, \quad B = \frac{1}{q - q^{-1}}(q^{-s}\mathbb{X}^{-1} - q^s\mathbb{X})\mathbb{Y}^{-1}, \quad C = \frac{1}{q - q^{-1}}(q^{-s}\mathbb{X} - q^s\mathbb{X}^{-1})\mathbb{Y}. \tag{2.10}$$

Such a realization may be also thought of as a q-deformation of the sl_2 algebra, when the corresponding generators are expressed in terms of differential operators, i.e.

$$S^z = y\frac{d}{dy}, \quad S^- = y^{-1}(y\frac{d}{dy} + \mathrm{s}), \quad S^+ = y(y\frac{d}{dy} - \mathrm{s}). \tag{2.11}$$

It is worth pointing out that the latter expressions for $\mathrm{s} = 0$, correspond to a $sl(2, \mathbb{R})$ realization, which has been employed in the context of high energy QCD. More precisely, it was shown in [36] that the high energy asymptotic behavior of the hadron-hadron scattering amplitude in QCD is described by the non-compact XXX Heisenberg chain with $\mathrm{s} = 0$ [37, 38].

In the present article the XXZ model will be examined for the special case where q is root of unity, i.e. $q^p = 1$, $q = e^{i\mu}$, $\mu = \frac{2k\pi}{p}$, where k, p integers. In this case the algebra admits a p dimensional representation, known as the cyclic representation [25]. More specifically, in addition to (2.9) one more restriction is applied so one may obtain a representation with no highest (lowest) weight

$$\mathbb{X}^p = \mathbb{Y}^p = 1$$

then the generators $\mathbb{X}$, $\mathbb{Y}$ may be expressed as p dimensional matrices

$$\mathbb{X} = \sum_{k=1}^{p} q^{-k} \, e_{kk}, \quad \mathbb{Y} = \sum_{k=1}^{p-1} e_{k \; k+1} + e_{p1}. \tag{2.12}$$

This realization is of particular interest and it has been extensively used for instance in the problem of Bloch electrons in a magnetic field, described by the Azbel-Hofstadter Hamiltonian [30, 31, 32]. In addition as argued in [39] the quantum Hall effect states may be also seen as cyclic representation of $\mathcal{U}_q(sl_2)$.

The XXZ model can be regarded as a universal model, since it gives rise to a whole class of associated integrable models. In what follows we shall briefly review how the lattice sine-Gordon and Liouville models are obtained in a natural way from the XXZ L matrix. Also the q harmonic realization will be obtained again from the generalized XXZ form. The generators $\mathbb{X}$ and $\mathbb{Y}$ may be associated with an infinite dimensional representation in terms of some lattice 'fields'. Then one may identify the L operator of the lattice sine–Gordon model [40]. More specifically set $\mathbb{Z}_n = \mathbb{Y}_n \, \mathbb{X}_n$, it is then clear that $\mathbb{X}_n \, \mathbb{Z}_n = q\mathbb{Z}_n \, \mathbb{X}_n$, where notice that for convenience we reintroduce the index n. Also set

$$\mathbb{X}_n \to e^{-i\Phi_n}, \quad \mathbb{Z}_n \to e^{i\Pi_n}, \quad \left[\Phi_n, \; \Pi_m\right] = i\mu\delta_{mn} \tag{2.13}$$

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$q^{s-\frac{1}{2}} = -im$. The parameter s of the representation is associated to the mass scale of the system. Also by multiplying by $-im\,\sigma^x$ (we are allowed to multiply with σ^x because this leaves the XXZ R matrix invariant) one obtains the lattice sine-Gordon L matrix [40]

$$L_{an}^{SG}(\lambda) = \begin{pmatrix} h_+(\Phi_n)e^{i\Pi_n} & -im\,2\sinh(\mu\lambda + i\Phi_n) \\ -im\,2\sinh(\mu\lambda - i\Phi_n) & h_-(\Phi)e^{-i\Pi_n} \end{pmatrix} \tag{2.14}$$

$$h_\pm(\Phi_n) = 1 + m^2 e^{\pm 2i\Phi_n + i\mu}.$$

As was argued in [41] by taking an appropriate massless limit a lattice version of the Liouville model is also recovered. Indeed by multiplying the sine–Gordon L operator with

$$L_{an}(\lambda) = g_a\,L_{an}^{SG}(\lambda)\,g_a^{-1}, \qquad g = diag\left((\frac{m}{i\alpha})^{\frac{1}{2}}, (\frac{m}{i\alpha})^{-\frac{1}{2}}\right) \tag{2.15}$$

where α will be associated to the spacing of the lattice and in the classical limit $\alpha \to 0$ (see below). In the quantum level is a finite number and for our purposes here we set $\alpha = -i$ (see also section 7). Consider also the following limiting process [41]

$$i\Phi_n \to i\Phi_n + c, \qquad \mu\lambda \to \mu\lambda + c, \qquad m \to 0, \qquad e^{-c} \to \infty, \qquad m^2 e^{-2c} \to \alpha^2 \tag{2.16}$$

one obtains the lattice Liouville L matrix

$$L_{an}^{Lv}(\lambda) = \begin{pmatrix} e^{i\Pi_n} & \alpha e^{-\mu\lambda - i\Phi_n} \\ 2\alpha\sinh(\mu\lambda - i\Phi_n) & h(\Phi_n)e^{-i\Pi_n} \end{pmatrix} \tag{2.17}$$

$$h(\Phi_n) = 1 + \alpha^2 e^{-2i\Phi_n + i\mu}.$$

The interesting observation is that the entailed L operator (2.17) has a non trivial spectral (λ) dependence a fact that allows the application of Bethe ansatz techniques for the derivation of the spectrum (see also [41]). The classical limit of the aforementioned L matrices (2.14), (2.17) provides the corresponding classical Lax operators satisfying the zero curvature condition, and giving rise to the classical equations of motion of the relevant models (for more details see also [40, 41]). Let us briefly review the connection between the quantum (lattice) versions and the classical sine-Gordon and Liouville models. Consider the following classical limit [40, 41], the spacing $\alpha \to 0$, set $\mu \to h\mu$ such that $\frac{1}{h}[,] \to \{,\}$, and

$$\Phi_n \to \frac{\beta}{2}\phi(x) - \frac{\pi}{2}, \qquad \Pi_n \to \alpha\frac{\beta}{4}\pi(x), \qquad -4im \to \alpha\tilde{m}, \qquad \mu\lambda \to u + \frac{i\pi}{2} \tag{2.18}$$

$\tilde{m}$ is the continuum mass and β corresponds to the coupling constant of the sine Gordon model, and for the Liouville model we set $\frac{\beta}{2} = 1$, following the normalization of [41]. Bearing in mind the expressions above (2.18) we obtain as $\alpha \to 0$

$$L(u) = 1 - \alpha U(u) + \mathcal{O}(\alpha^2) \tag{2.19}$$

then the quantities $U(u)$ and $V(u)$, written below provide a Lax pair for the classical counterparts of the lattice sine-Gordon and Liouville models. More precisely for the sine Gordon model

$$U(u) = \frac{1}{2}\begin{pmatrix} -i\frac{\beta}{2}\pi(x) & -\tilde{m}\sinh(u + i\frac{\beta}{2}\phi(x)) \\ \tilde{m}\sinh(u - i\frac{\beta}{2}\phi(x)) & i\frac{\beta}{2}\pi(x) \end{pmatrix},$$

$$V(u) = \frac{1}{2}\begin{pmatrix} -i\frac{\beta}{2}\phi'(x) & -\tilde{m}\cosh(u + i\frac{\beta}{2}\phi(x)) \\ \tilde{m}\cosh(u - i\frac{\beta}{2}\phi(x)) & i\frac{\beta}{2}\phi'(x) \end{pmatrix} \tag{2.20}$$

whereas for the Liouville model the Lax pair reads

$$U(u) = \frac{1}{2} \begin{pmatrix} -i\pi(x) & -2e^{-u-i\phi(x)} \\ 4\sinh(u - i\phi(x)) & i\pi(x) \end{pmatrix}, \quad V(u) = \frac{1}{2} \begin{pmatrix} -i\phi'(x) & 2e^{-u-i\phi(x)} \\ 4\cosh(u - i\phi(x)) & i\phi'(x) \end{pmatrix} \tag{2.21}$$

The Lax pair satisfies the zero curvature condition

$$\dot{U} - V' + \left[U,\ V \right] = 0 \tag{2.22}$$

which leads to the corresponding equations of motions i.e.

$$\text{sine-Gordon model:} \quad \ddot{\phi}(x) - \phi''(x) + \frac{\tilde{m}^2}{\beta}\sin(\beta\phi(x)) = 0$$

$$\text{Liouville model:} \quad \ddot{\phi}(x) - \phi''(x) - 4ie^{-2i\phi(x)} = 0. \tag{2.23}$$

Similar limiting process to (2.16) leads to the q-harmonic oscillator L matrix starting from (2.5), (2.10). In fact, by simply multiplying the Liouville L matrix with an anti-diagonal matrix and bearing also in mind (2.13) we obtain the following

$$L_{an}(\lambda) = \begin{pmatrix} e^{\mu\lambda}V_n - e^{-\mu\lambda}V_n^{-1} & a_n^+ \\ a_n & -e^{-\mu\lambda}V_n \end{pmatrix} \tag{2.24}$$

where the operators a_n, a_n^+, V_n are expressed in terms of $\mathbb{X}_n$, $\mathbb{Y}_n$ as

$$V_n = \mathbb{X}_n, \quad a_n^+ = (\mathbb{X}_n^{-1} - q\mathbb{X}_n)\mathbb{Y}_n^{-1}, \quad a_n = \mathbb{Y}_n\,\mathbb{X}_n \tag{2.25}$$

and they satisfy the q harmonic oscillator algebra i.e.

$$a_n^+\,a_n = 1 - qV_n^2, \quad a_n\,a_n^+ = 1 - q^{-1}V_n^2. \tag{2.26}$$

The latter may be also seen as q deformation of the Discrete-Self-Trapping (DST) model [42]. The DST equation introduced in [43] to describe the non-linear dynamical behaviour of small molecules (ammonia, acetylene), and big molecules as well (acetanilide). Also the integrability properties of the DST model with two or more degrees of freedom were studied in a series of articles [34, 44].

Finally, the simplest q deformed L matrix is obtained from (2.14) —written in terms on the algebraic objects X_n, Z_n (see (2.13))— via a simple limit

$$\mu\lambda \to \mu\lambda + b, \quad e^b \to \infty, \quad -ime^b \to 1 \tag{2.27}$$

then the corresponding L matrix becomes (see also [31])

$$L_{an}(\lambda) = \begin{pmatrix} \mathbb{Z}_n & e^{\mu\lambda}\mathbb{X}_n^{-1} \\ e^{\mu\lambda}\mathbb{X}_n & \mathbb{Z}_n^{-1} \end{pmatrix} \tag{2.28}$$

and this L matrix was used for the construction of lattice versions of the KdV system (for more details on the lattice KdV see also [45, 46, 47]).

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3 The open transfer matrix

As is well known to construct the transfer matrix associated to an open spin chain one needs in addition to the R (L) matrix one more fundamental object, namely the K matrix, which is proportional to the physical boundary S matrix, acting on $\mathbb{V}$, and is a solution of the reflection equation [6],

$$R_{12}(\lambda_1 - \lambda_2) \, K_1(\lambda_1) \, R_{21}(\lambda_1 + \lambda_2) \, K_2(\lambda_2) = K_2(\lambda_2) \, R_{12}(\lambda_1 + \lambda_2) \, K_1(\lambda_1) \, R_{21}(\lambda_1 - \lambda_2). \tag{3.1}$$

As proposed in [28, 48] an effective way of finding solutions of the reflection equations (3.1) is to exploit certain algebraic structures such as (boundary) Temperley–Lieb algebras [48]–[52].

Let us recall how one may express the XXZ R and K matrices, solutions of the Yang-Baxter and reflection equations respectively, in terms of the generators of the boundary Temperley-Lieb (blob) algebra. First define the blob algebra $b_N(q, Q)$, which is a quotient of the affine Hecke algebra [50], with generators $\mathbb{U}_1, \mathbb{U}_2, ..., \mathbb{U}_{N-1}$ and $\mathbb{U}_0$, and relations:

$$
\begin{aligned}
\mathbb{U}_l \, \mathbb{U}_l &= \delta \, \mathbb{U}_l, \quad \mathbb{U}_0 \, \mathbb{U}_0 = \delta_0 \, \mathbb{U}_0 \\
\mathbb{U}_{l\pm 1} \, \mathbb{U}_l \, \mathbb{U}_{l\pm 1} &= \mathbb{U}_{l\pm 1}, \quad \mathbb{U}_1 \, \mathbb{U}_0 \, \mathbb{U}_1 = \gamma \, \mathbb{U}_1 \\
\left[\mathbb{U}_l, \, \mathbb{U}_k \right] &= 0, \quad |l - k| \neq 1
\end{aligned}
\tag{3.2}
$$

$\delta = -(q + q^{-1})$, $q = e^{i\mu}$, and δ_0, γ are constants depending on q and $Q = e^{im\mu}$. Physically the constants q and Q play the role of the bulk and boundary parameter respectively of the open spin chain, which will be constructed in the following.

The generators $\mathbb{U}_l$, $l \in \{1, .., N-1\}$, $\mathbb{U}_0$ of the blob algebra are given in the spin $\frac{1}{2}$ XXZ representation, i.e. let the tensor representation $h : b_N(q, Q) \to \text{End}((\mathbb{C}^2)^{\otimes N})$ such that (see also [51])

$$
\begin{aligned}
h(\mathbb{U}_l) &= 1 \otimes \ldots \otimes
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -q & 1 & 0 \\
0 & 1 & -q^{-1} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\otimes \ldots \otimes 1, \\
h(\mathbb{U}_0) &=
\begin{pmatrix}
-Q & q^{\theta} \\
q^{-\theta} & -Q^{-1}
\end{pmatrix}
\otimes \ldots \otimes 1
\end{aligned}
\tag{3.3}
$$

with $h(\mathbb{U}_l)$ acting non-trivially on $\mathbb{V}_l \otimes \mathbb{V}_{l+1}$, (where $\mathbb{V} = \mathbb{C}^2$) and $h(\mathbb{U}_0)$ acting on $\mathbb{V}_1$. Note that $h(\mathbb{U}_l)$ $l \in \{1, \ldots, N-1\}$ are actually the XXZ representations of the Temperley-Lieb algebra generators [49, 52]. The notation here is slightly modified compared to [28]. Also, for this representation we find: $\delta_0 = -(Q + Q^{-1})$ and $\gamma = qQ + q^{-1}Q^{-1}$.

As argued in [28] tensor representations of the blob algebra provide solutions of the reflection equation. Hence a solution of the reflection equation (3.1) may be written in terms of the blob algebra generator $h(\mathbb{U}_0)$ (3.3) as

$$K_1^{(b)}(\lambda) = 2 \sinh \mu \left(\lambda - \frac{im}{2} - i\zeta \right) \cosh \mu \left(\lambda - \frac{im}{2} + i\zeta \right) \mathbb{I} + \sinh(2\mu\lambda) \, h(\mathbb{U}_0), \tag{3.4}$$

ζ is an arbitrary constant. The XXZ spin $\frac{1}{2}$ R matrix (2.2) may be also written in terms of the representation $h(\mathbb{U}_l)$, [53], i.e.

$$R_{12}(\lambda) = \mathcal{P}_{12}\Big(\sinh\mu(\lambda+i)\ \mathbb{I} + \sinh(\mu\lambda)\ h(\mathbb{U}_1) \Big) \tag{3.5}$$

where $\mathcal{P}$ is the permutation operator, such that $\mathcal{P}(a \otimes b) = b \otimes a$.

The matrix obtained from the blob algebra (3.4) coincides with the K matrix found in [54, 55] (written in the homogeneous gradation) i.e.

$$K(\lambda) = \begin{pmatrix} \sinh\mu(-\lambda+i\xi)\ e^{\mu\lambda} & q^\theta \kappa \sinh(2\mu\lambda) \\ q^{-\theta}\kappa \sinh(2\mu\lambda) & \sinh\mu(\lambda+i\xi)\ e^{-\mu\lambda} \end{pmatrix} \tag{3.6}$$

subject to the following identifications

$$\frac{e^{-i\mu\xi}}{2\kappa} = \sinh(im\mu), \quad \frac{e^{i\mu\xi}}{2\kappa} = -\sinh(2i\mu\zeta). \tag{3.7}$$

The K matrix (3.4), (3.6) is written in the homogeneous gradation. To obtain the matrix in the principal gradation it is necessary to perform a simple gauge transformation, i.e.

$$K^{(p)}(\lambda) = \mathcal{V}(\lambda)\ K^{(h)}(\lambda)\ \mathcal{V}(\lambda) \tag{3.8}$$

where $\mathcal{V}$ is given by (2.3).

Another representation that leads eventually to upper (lower) triangular K matrices follows essentially if one considers the limit $q^\theta,\ Q \to \infty$, then the representation of the boundary element (3.3) reduces to:

$$h(\mathbb{U}_0) = \begin{pmatrix} -Q & q^\theta \\ 0 & 0 \end{pmatrix} \otimes \ldots \otimes 1. \tag{3.9}$$

Indeed the matrices $h(\mathbb{U}_l)$ (3.3) and $h(\mathbb{U}_0)$ (3.9) form a representation of the blob algebra with $\delta_0 = -Q$ and $\gamma = q\,Q$. The corresponding K matrix then is

$$K(\lambda) = \begin{pmatrix} \sinh\mu(-\lambda+i\xi)\ e^{\mu\lambda} & q^\theta\kappa\sinh(2\mu\lambda) \\ 0 & \sinh\mu(\lambda+i\xi)\ e^{-\mu\lambda} \end{pmatrix} \tag{3.10}$$

where in this case the relations among the boundary parameters become

$$\frac{e^{-i\mu\xi}}{\kappa} = e^{im\mu}, \quad \frac{e^{i\mu\xi}}{2\kappa} = -\sinh(2i\mu\zeta). \tag{3.11}$$

It is clear that by considering $q^{-\theta} \to \infty$ one recovers a lower triangular matrix. In fact, such upper (lower) triangular K matrices may be thought of as the result of a very particular gauge transformation of the type (2.15), where the mass parameter $m \propto q^\theta$ becomes very big (or very small).

Finally, to construct the generating function of the conserved commuting quantities associated to an open spin chain, one needs two solutions $K^\pm(\lambda)$ of the reflection equation (3.1). Indeed, the transfer matrix $t(\lambda)$ associated to an open spin chain of N sites [7] reads

$$t(\lambda) = \mathrm{tr}_0 \left\{ K_0^+(\lambda)\ T_0(\lambda)\ K_0^-(\lambda)\ \hat{T}_0(\lambda) \right\} = \mathrm{tr}_0 \left\{ K_0^+(\lambda)\ \mathcal{T}_0(\lambda) \right\} \tag{3.12}$$

where $\mathcal{T}$ is a solution of the reflection equation (3.1) and

$$T_0(\lambda) = L_{01}(\lambda - i\Theta)L_{02}(\lambda - i\Theta) \cdots L_{0N}(\lambda - i\Theta) \quad \hat{T}_0(\lambda) = \hat{L}_{0N}(\lambda + i\Theta) \cdots \hat{L}_{02}(\lambda + i\Theta)\hat{L}_{01}(\lambda + i\Theta) \quad (3.13)$$

L is one of the matrices introduced in section 2 and $\hat{L}(\lambda) = L^{-1}(-\lambda)$. As usually we keep only the index 0, associated to the 'auxiliary' space, whereas the indices $1, \ldots, N$, corresponding to the 'quantum' spaces are suppressed. The parameter Θ is called inhomogeneity, and in principle one could attach a different one at each site of the spin chain. Also $K^+(\lambda) = M \; K(-\lambda - i)$, K is any solution of the reflection equation, and M is defined as:

$$\begin{aligned} M &= \mathbb{I}, & \text{Principal gradation} \\ M &= diag(q, \; q^{-1}), & \text{Homogeneous gradation.} \end{aligned} \quad (3.14)$$

K^- is a c-number solution (3.6) of the reflection equation as well. The transfer matrix (3.12) provides a family of commuting operators [7], i.e.

$$\Big[t(\lambda), \; t(\lambda') \Big] = 0. \quad (3.15)$$

At this stage the quantity (3.12), (3.13) is a purely algebraic construction. It only acquires a physical meaning describing a spin chain system once particular representations are chosen to act on the quantum spaces.

4 Hamiltonian of a spin chain coupled to a dynamical boundary

Already in the introduction the general form of a Hamiltonian with dynamical boundary term was presented (1.1). We focus here in a special case of the algebraic transfer matrix (3.12), (3.13) described in the previous section. More precisely, we consider the situation where the spin $\frac{1}{2}$ representation is assigned at each quantum space ($L \mapsto R$, set also $\Theta = 0$), and the right end of the chain is coupled to dynamical degrees of freedom, associated to the $\mathcal{U}_q(sl_2)$, and the q harmonic oscillator algebras i.e.

$$t(\lambda) = tr_0 \Big\{ K_0^+(\lambda) \; R_{01}(\lambda) \ldots R_{0N}(\lambda) \; K_0^-(\lambda) \; \hat{R}_{0N}(\lambda) \ldots \hat{R}_{01}(\lambda) \Big\} \quad (4.1)$$

where R is the XXZ matrix (2.2), and now the K^- matrix is expressed in a factorized form [7, 26, 27, 20, 28, 29, 24]

$$K_0^-(\lambda) = L_{0\mathrm{d}}(\lambda - i\Theta_0) \; K_0(\lambda) \; \hat{L}_{0\mathrm{d}}(\lambda + i\Theta_0) \quad (4.2)$$

the index d characterizes the extra 'dynamical space' attached to the right boundary of the chain, K is a c-number solution of the reflection equation (3.6) with parameters ξ^-, κ^-, θ^-, L being the matrix given by (2.5) or (2.24), and $K^+ = MK'(-\lambda - i)$ where K' is again a c-number solution of the reflection equation of the general type (3.6), but with parameters ξ^+, κ^+, θ^+. Note that the dynamical space may be thought of as an extra site of the spin chain, but here it is preferable to treat it as a dynamical system attached to the boundary. It will be instructive for future reference to provide at this point explicit expressions for the dynamical matrix $\mathcal{M}$ appearing in the Hamiltonian (1.1), and due to the

dynamical nature of the boundary the entries of $\mathcal{M}$ in (1.1) will be written in terms of the generators of the $\mathcal{U}_q(sl_2)$ and q harmonic oscillator algebras as we will see below. In general the Hamiltonian is defined as proportional to $\frac{d}{d\lambda}t(\lambda)|_{\lambda=0}$, in particular we choose to normalize as (we also set $\Theta = 0$):

$$\mathcal{H} = -\frac{(\sinh(i\mu))^{-2N+1}\mathcal{F}^{-1}}{4\mu\sinh(i\mu\xi^-)}\left(tr_0 K_0^+(\lambda)\right)^{-1} \frac{d}{d\lambda}t(\lambda)|_{\lambda=0}, \tag{4.3}$$

where $\mathcal{F} = L_{0d}(-i\Theta_0)\,\hat{L}_{0d}(i\Theta_0)$, and for:

(I) the $\mathcal{U}_q(sl_2)$ case, $\mathcal{F} = w - \cosh i\mu(1 + 2\Theta_0)$ with w being the Casimir of $\mathcal{U}_q(sl_2)$

$$w = q\,A^2 + q^{-1}D^2 + (q - q^{-1})^2 B\,C = q^{-1}\,A^2 + qD^2 + (q - q^{-1})^2 C\,B. \tag{4.4}$$

(II) the q-harmonic oscillator, $\mathcal{F} = 1 - q^{1+2\Theta_0}$.

Then having in mind that $R(0) = \hat{R}(0) = \sinh i\mu\,\mathcal{P}$, $K(0) = \sinh(i\mu\xi^-)\,\mathbb{I}$ and also define

$$H_{kl} = -\frac{1}{2\mu}\frac{d}{d\lambda}(\mathcal{P}_{kl}\,R_{kl}(\lambda)) \tag{4.5}$$

we may rewrite the Hamiltonian as

$$\mathcal{H} = \sum_{l=1}^{N-1} H_{l\,l+1} - \frac{\sinh(i\mu)\,\mathcal{F}^{-1}}{4\mu\sinh(i\mu\xi^-)}\left(\frac{d}{d\lambda}K_N^-(\lambda)\right)\Big|_{\lambda=0} + \frac{tr_0\,K_0^+(0)\,H_{10}}{tr_0\,K_0^+(0)}. \tag{4.6}$$

The first term in (4.6) gives rise to the bulk spin-spin interaction between first neighbours appearing in (1.1), whereas the second term gives rise to $\mathcal{M}$, and the third one to the last three terms of (1.1). We shall provide at this point the explicit values of the constants c_i of (1.1). It is straightforward to compute from (4.6) that:

$$c_1 = -\frac{\sinh(i\mu)\cosh(i\mu\xi^+)}{4\sinh(i\mu\xi^+)}, \qquad c_2 = -\frac{\sinh(i\mu)\kappa^+ q^{-\theta^+}}{4\sinh(i\mu\xi^+)}, \qquad c_3 = -\frac{\sinh(i\mu)\kappa^+ q^{\theta^+}}{4\sinh(i\mu\xi^+)}, \tag{4.7}$$

where the parameters ξ^+, κ^+, θ^+ are apparently associated to the left boundary K^+. To summarize the Hamiltonian is finally written exactly as in (1.1)

$$\begin{aligned}
\mathcal{H} &= -\frac{1}{4}\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \cosh(i\mu)\,\sigma_i^z\sigma_{i+1}^z\right) - \frac{N}{4}\,\cosh(i\mu) + \mathcal{M}_N \\
&\quad - \frac{\sinh(i\mu)\cosh(i\mu\xi^+)}{4\sinh(i\mu\xi^+)}\sigma_1^z - \frac{\sinh(i\mu)\kappa^+ q^{-\theta^+}}{4\sinh(i\mu\xi^+)}\sigma_1^+ - \frac{\sinh(i\mu)\kappa^+ q^{\theta^+}}{4\sinh(i\mu\xi^+)}\sigma_1^-
\end{aligned} \tag{4.8}$$

Our objective in what follows is to give explicit expressions for the matrix $\mathcal{M}$ occurring as a boundary contribution in the Hamiltonian above due to the second term in (4.6).

(I) *The $\mathcal{U}_q(sl_2)$ algebra:* The entries of $\mathcal{M}$ 'act' to the 'dynamical space' which in this case is associated to a copy of $\mathcal{U}_q(sl_2)$, hence they are expressed in terms of the $\mathcal{U}_q(sl_2)$ generators as

$$\mathcal{M}_{11} = -\frac{\sinh(i\mu)\,\mathcal{F}^{-1}}{4\sinh(i\mu\xi^-)}\left[-\cosh(i\mu\xi^-)\left(2q\,A^2 + 2q^{-1}D^2 - w - q^{1+2\Theta_0} - q^{-1-2\Theta_0}\right)\right.$$

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$$+ \ 2\kappa^-(q-q^{-1})\Big(q^{-\theta^-}B(q^{1+\Theta_0}A - q^{-1-\Theta_0}D) + q^{\theta^-}(q^{-\Theta_0}A - q^{\Theta_0}D)C\Big) + 2\sinh(i\mu\xi^-)\Big(qA^2 - q^{-1}D^2\Big)\Big]$$

$$\mathcal{M}_{22} = -\frac{\sinh(i\mu)\ \mathcal{F}^{-1}}{4\sinh(i\mu\xi^-)}\Big[\cosh(i\mu\xi^-)\Big(2q^{-1}\ A^2 + 2qD^2 - w - q^{1+2\Theta_0} - q^{-1-2\Theta_0}\Big)$$

$$+ \ 2\kappa^-(q-q^{-1})\Big(q^{\theta^-}C(q^{1+\Theta_0}D - q^{-1-\Theta_0}A) + q^{-\theta^-}(q^{-\Theta_0}D - q^{\Theta_0}A)B\Big) + 2\sinh(i\mu\xi^-)\Big(qD^2 - q^{-1}A^2\Big)\Big]$$

$$\mathcal{M}_{12} = -\frac{\sinh(i\mu)\ \mathcal{F}^{-1}}{4\sinh(i\mu\xi^-)}\Big[2\cosh(i\mu\xi^-)(q-q^{-1})(q^{\Theta_0}D - q^{-\Theta_0}A)B + 2\sinh(i\mu\xi^-)(q-q^{-1})(q^{-\Theta_0}A + q^{\Theta_0}D)B$$

$$+ \ 2\kappa^-\Big(q^{-\theta^-}(q-q^{-1})^2B^2 + q^{\theta^-}(q+q^{-1} - q^{1+2\Theta_0}D^2 - q^{-1-2\Theta_0}A^2)\Big)\Big]$$

$$\mathcal{M}_{21} = -\frac{\sinh(i\mu)\ \mathcal{F}^{-1}}{4\sinh(i\mu\xi^-)}\Big[2\cosh(i\mu\xi^-)(q-q^{-1})(q^{-\Theta_0}D - q^{\Theta_0}A)C + 2\sinh(i\mu\xi^-)(q-q^{-1})(q^{\Theta_0}A + q^{-\Theta_0}D)C$$

$$+ \ 2\kappa^-\Big(q^{\theta^-}(q-q^{-1})^2C^2 + q^{-\theta^-}(q+q^{-1} - q^{-1-2\Theta_0}D^2 - q^{1+2\Theta_0}A^2)\Big)\Big]. \tag{4.9}$$

(II) *The q harmonic oscillator algebra*: In this case the entries of $\mathcal{M}$ belong to the q harmonic oscillator algebra (2.26) generated by V, a, a^+

$$\mathcal{M}_{11} = -\frac{\sinh(i\mu)\ \mathcal{F}^{-1}}{4\sinh(i\mu\xi^-)}\Big[\cosh(i\mu\xi^-)(1 + q^{1+2\Theta_0} - 2qV^2) + 2\sinh(i\mu\xi^-)qV^2$$

$$+2\kappa^-\Big(q^{-\theta^-+1}a^+V + q^{\theta^-}(q^{-\Theta_0}V - q^{\Theta_0}V^{-1})a\Big)\Big]$$

$$\mathcal{M}_{22} = -\frac{\sinh(i\mu)\ \mathcal{F}^{-1}}{4\sinh(i\mu\xi^-)}\Big[-\cosh(i\mu\xi^-)(1 + q^{1+2\Theta_0} - 2q^{-1}V^2) - 2\sinh(i\mu\xi^-)q^{-1}V^2$$

$$+2\kappa^-\Big(-q^{-\theta^-}q^{\Theta_0}Va^+ - q^{\theta^-}a(q^{-1-2\Theta_0}V - q^{1+2\Theta_0}V^{-1})\Big)\Big]$$

$$\mathcal{M}_{12} = -\frac{\sinh(i\mu)\ \mathcal{F}^{-1}}{4\sinh(i\mu\xi^-)}\Big[-2\cosh(i\mu\xi^-)(q^{-\Theta_0}V - q^{\Theta_0}V^{-1})a^+ + 2\sinh(i\mu\xi^-)(q^{-\Theta_0}V + q^{\Theta_0}V^{-1})a^+$$

$$+2\kappa^-\Big(q^{\theta^-}(q+q^{-1} - q^{1+2\Theta_0}V^{-2} - q^{-1-2\Theta_0}V^2) + q^{-\theta^-}(a^+)^2\Big)\Big]$$

$$\mathcal{M}_{21} = -\frac{\sinh(i\mu)\ \mathcal{F}^{-1}}{4\sinh(i\mu\xi^-)}\Big[-2\cosh(i\mu\xi^-)q^{\Theta_0}Va + 2\sinh(i\mu\xi^-)q^{\Theta_0}Va + 2\kappa^-\Big(q^{\theta^-}a^2 - q^{-\theta^-}q^{-1+2\Theta_0}V^2\Big)\Big]. \tag{4.10}$$

As expected the entries of the matrix $\mathcal{M}$, due to right dynamical boundary, are expressed in terms of the generators of the two main algebras i.e. $\mathcal{U}_q(sl_2)$ and the q harmonic oscillator. These generators occur naturally in the expressions above due to the presence of L_{0d}, $\hat{L}_{0d}$ in the dynamical boundary we are considering (4.2). To find the spectrum and the eigenstates of such a Hamiltonian one has to fix the representation at the boundary, and this will be done in section 6 where the cyclic representation (2.12) is considered for both $\mathcal{U}_q(sl_2)$ and the q harmonic oscillator.

5 The boundary non-local charges

The so called boundary non-local charges [56]–[59] will be now introduced. Such charges are going to play an essential role in the derivation of the spectrum as we shall see in a subsequent section. They are obtained via the standard procedure, by considering the asymptotics of the operator $\mathcal{T}$ as $\mu\lambda \to \infty$ (see

also [57, 58]). Note that in what follows the L, $\hat{L}$ and $K^{\pm}$ matrices are written in the principal gradation.

(I) *The XXZ and lattice sine Gordon models*: The asymptotics of $L(\lambda - \Theta)$, $\hat{L}(\lambda + \Theta)$ as $\mu\lambda \to \infty$, bearing in mind the evaluation homomorphism (A.5), is given as ($\mu\Theta$ is considered finite)

$$L_{0n}(\mu\lambda \to \infty) \propto \begin{pmatrix} k_{1,n} & \\ & k_{2,n} \end{pmatrix} + e^{-\mu\lambda} w \begin{pmatrix} & f_{1,n} \\ f_{2,n} & \end{pmatrix} + \dots,$$

$$\hat{L}_{0n}(\mu\lambda \to \infty) \propto \begin{pmatrix} k_{1,n} & \\ & k_{2,n} \end{pmatrix} + e^{-\mu\lambda} \hat{w} \begin{pmatrix} & e_{2,n} \\ e_{1,n} & \end{pmatrix} + \dots \tag{5.1}$$

with $k_2 = k_1^{-1}$. The subscript n simply denotes that the corresponding objects are associated to the nth site ('quantum space') of the chain,

$$w = 2 \ \sinh(i\mu) \ q^{-\frac{1}{2}+\Theta'}, \quad \hat{w} = 2 \ \sinh(i\mu) \ q^{-\frac{1}{2}-\Theta'}, \quad \Theta' = \Theta + \frac{1}{2}. \tag{5.2}$$

By also considering K^{-}, being of the form (3.6) with boundary parameter ξ^{-}, κ^{-}, θ^{-}, as $\mu\lambda \to \infty$ (3.6) (the constants ξ^{-}, κ^{-}, θ^{-} are considered to be finite)

$$K^{-}(\mu\lambda \to \infty) \propto \begin{pmatrix} q^{\theta^{-}} \kappa^{-} & \\ q^{-\theta^{-}} \kappa^{-} & \end{pmatrix} + e^{-\mu\lambda} \begin{pmatrix} -e^{-i\mu\xi^{-}} & \\ & e^{i\mu\xi^{-}} \end{pmatrix} + \dots \tag{5.3}$$

one may formulate the corresponding behavior of $\mathcal{T}$ (3.12), namely

$$\mathcal{T}(\mu\lambda \to \infty) \propto \begin{pmatrix} q^{\theta^{-}} & \\ & q^{-\theta^{-}} \end{pmatrix} + 2\sinh(i\mu) \ e^{-\mu\lambda} \begin{pmatrix} Q_1^{(N)} & \\ & Q_2^{(N)} \end{pmatrix} + \dots \tag{5.4}$$

where the non-local charges $Q_i^{(N)}$, $i \in \{1, \ 2\}$ are given by the following expressions [56]–[59]

$$Q_i^{(N)} = q^{-\frac{1}{2}-\Theta_i} K_i^{(N)} E_i^{(N)} + q^{\frac{1}{2}+\Theta_i} K_i^{(N)} F_i^{(N)} + x_i (K_i^{(N)})^2, \quad i \in \{1, \ 2\}. \tag{5.5}$$

$K_i^{(N)}$, $E_i^{(N)}$ and $F_i^{(N)}$, provide N coproducts of the quantum Kac–Moody algebra $\mathcal{U}_q(\widehat{sl_2})$ (see also Appendix)

$$K_i^{(N)} = \Delta^{(N)}(k_i), \quad E_i^{(N)} = \Delta^{(N)}(e_i), \quad F_i^{(N)} = \Delta^{(N)}(f_i), \quad i \in \{1, \ 2\} \tag{5.6}$$

and the constants appearing in (5.5) are:

$$x_1 = -\frac{e^{-i\mu\xi^{-}}}{2\kappa^{-} \sinh(i\mu)}, \quad x_2 = \frac{e^{i\mu\xi^{-}}}{2\kappa^{-} \sinh(i\mu)}, \quad \Theta_1 = \Theta' - \theta^{-}, \quad \Theta_2 = \Theta' + \theta^{-}. \tag{5.7}$$

It is clear that in the case of an upper triangular K^{-} matrix the boundary non-local charges follow immediately from the expressions above, given that the term proportional to $q^{-\theta^{-}}$ is omitted. Analogous expressions are obtained for a lower triangular matrix, and in this case the term proportional to $q^{\theta^{-}}$ is omitted in the expressions (5.5).

It is worth elaborating a bit further on the lattice sine–Gordon. Our aim is to provide explicit expressions of the lattice topological charge [40] and the fractional-spin integrals of motion for the lattice

sine Gordon model. To achieve this we recall that locally the L matrix (2.14) lacks a highest (lowest) weight, and as a consequence the algebraic Bethe ansatz formulation cannot be applied. However, it was proposed in [40] that one may consider a $2N$ site spin chain and locally can deal with a new L matrix constructed as a combination of two successive L matrices. Here we are dealing with a system with open boundaries, and we also need to define the corresponding $\hat{L}$ matrix. Indeed we define the following objects

$$\mathbb{L}_{0n}(\lambda) = L_{0\ 2n-1}(\lambda)\ L_{0\ 2n}(\lambda), \quad \hat{\mathbb{L}}_{0n}(\lambda) = \hat{L}_{0\ 2n}(\lambda)\ \hat{L}_{0\ 2n-1}(\lambda) \tag{5.8}$$

where the subscripts $2n$, $2n-1$ in the L operators denote the corresponding site in the one dimensional lattice. Then by keeping the lowest order terms in the $\mu\lambda \to \infty$ expansion the $\mathbb{L}$, $\hat{\mathbb{L}}$ matrices reduce to the same form as in (5.1). In this case in particular k_i, e_i, f_i are given by the following expressions:

$$k_{1,n} = e^{-i(\Phi_{2n}-\Phi_{2n-1})}, \qquad k_{2,n} = k_{1,n}^{-1}$$

$$e_{1,n} = -\frac{im}{q-q^{-1}}(h_+(\Phi_{2n})e^{i\Pi_{2n}}e^{i\Phi_{2n-1}} + h_-(\Phi_{2n-1})e^{-i\Pi_{2n-1}}e^{i\Phi_{2n}}),$$

$$f_{1,n} = -\frac{im}{q-q^{-1}}(h_+(\Phi_{2n-1})e^{i\Pi_{2n-1}}e^{i\Phi_{2n}} + h_-(\Phi_{2n})e^{-i\Pi_{2n}}e^{i\Phi_{2n-1}}),$$

$$e_{2,n} = -\frac{im}{q-q^{-1}}(h_-(\Phi_{2n})e^{-i\Pi_{2n}}e^{-i\Phi_{2n-1}} + h_+(\Phi_{2n-1})e^{i\Pi_{2n-1}}e^{-i\Phi_{2n}}),$$

$$f_{2,n} = -\frac{im}{q-q^{-1}}(h_+(\Phi_{2n})e^{i\Pi_{2n}}e^{-i\Phi_{2n-1}} + h_-(\Phi_{2n-1})e^{-i\Pi_{2n-1}}e^{-i\Phi_{2n}}). \tag{5.9}$$

In the context of the lattice sine–Gordon the N coproduct (see also Appendix A) $K_1^{(N)}$ plays the role of the lattice topological charge, whereas the coproducts $E_1^{(N)}$, $F_1^{(N)}$ play the role of the lattice fractional-spin integrals of motions, i.e. the lattice analogues of the classical expressions appearing in [60], generating $\mathcal{U}_q(\widehat{sl_2})$.

(II) *The q harmonic oscillator*: The asymptotic expansion of $L(\lambda - \Theta)$ and $\hat{L}(\lambda + \Theta)$ as $\mu\lambda \to \infty$ gives rise to the following matrices

$$L_{0n}(\mu\lambda \to \infty) \propto \begin{pmatrix} V_n & \\ & 0 \end{pmatrix} + w'e^{-\mu\lambda}\begin{pmatrix} & a_n^+ \\ a_n & \end{pmatrix} + \dots,$$

$$\hat{L}_{0n}(\mu\lambda \to \infty) \propto \begin{pmatrix} V_n & \\ & V_n^{-1} \end{pmatrix} + \hat{w}'e^{-\mu\lambda}\begin{pmatrix} & a_n^+ \\ a_n & \end{pmatrix} + \dots \tag{5.10}$$

$$w' = q^{-\frac{1}{2}+\Theta'}, \qquad \hat{w}' = q^{-\frac{1}{2}-\Theta'}. \tag{5.11}$$

Finally forming the asymptotics of $\mathcal{T}$, and bearing in mind (5.10), (5.3), we obtain the corresponding boundary non-local charges for the q harmonic oscillator

$$\mathcal{T}(\mu\lambda \to \infty) \propto \begin{pmatrix} q^{\theta^-} & \\ & \end{pmatrix} + e^{-\mu\lambda}\begin{pmatrix} Q_1^{(N)} & \\ & Q_2^{(N)} \end{pmatrix} + \dots \tag{5.12}$$

and the explicit expressions of the boundary charges, which are new are given by,

$$Q_1 = q^{-\frac{1}{2}+\Theta_1}A^{+(N)}V^{(N)} + q^{-\frac{1}{2}-\Theta_1}V^{(N)}\hat{A}^{(N)} - \frac{e^{-i\mu\xi^-}}{\kappa^-}(V^{(N)})^2, \quad Q_2^{(N)} = q^{-\frac{1}{2}+\Theta_2}A^{(N)}(V^{(N)})^{-1} \tag{5.13}$$

where we define

$$A^{+(N)} = \bigotimes_{n=1}^{N-1} V_n \, a_N^+, \quad A^{(N)} = a_1 \bigotimes_{n=2}^{N} V_n, \quad V^{(N)} = \bigotimes_{n=1}^{N} V_n$$

$$\hat{A}^{(N)} = \sum_{n=1}^{N} V \otimes \ldots \otimes V \otimes a_n \otimes V^{-1} \ldots \otimes V^{-1}. \tag{5.14}$$

Similarly if one considers an upper (lower) triangular K^- matrix then the corresponding boundary charges are obtained by ignoring the terms proportional to $q^{-\theta^-}$ (q^{θ^-}) exactly as in the previous case.

It was shown in [58, 59] that depending on the choice of the left boundary either $Q_1^{(N)}$ or $Q_2^{(N)}$ or a combination of the two is conserved. For instance for the special case where the left boundary (principal gradation) is

$$K^+(\lambda) = diag(e^{\mu(\lambda+i)}, \; e^{-\mu(\lambda+i)}) \tag{5.15}$$

the charge $Q_1^{(N)}$ is a conserved quantity [58]. Actually a stronger statement was shown in [58], i.e. the charge $Q_1^{(N)}$ turns out to be also the centralizer of the boundary Temperley–Lieb (blob) algebra in the spin $\frac{1}{2}$ XXZ representation. This immediately implies the commutation of the corresponding Hamiltonian, written in terms of the blob algebra generators, with $Q_1^{(N)}$. The derived non-local charges will turn out to play a crucial role in the identification of the spectrum of the corresponding models, hence they are not only of mathematical but of physical significance as well. Their relevance to the spectrum will be examined in detail in the subsequent section.

6 Diagonalization of the transfer matrix and Bethe ansatz

6.1 The pseudo-vacuum

To identify the spectrum and formulate the Bethe ansatz equations of the models under consideration we shall apply the method employed in [19, 33]. In the present section we deal with the XXZ model and the q harmonic oscillator, in the cyclic representation (2.12), with non-diagonal boundaries. The lattice sine–Gordon and Liouville models with diagonal boundaries only will be treated in a subsequent section. Let us give a brief account of the method, however for a more detailed analysis on the subject we refer the reader to [19]. One first introduces suitable gauge transformations known also as Darboux matrices (see e.g. [34])

$$M_n(\lambda) = \Big(X_n(\lambda), \; Y_n(\lambda)\Big), \quad \bar{M}_n(\lambda) = \Big(X_{n+1}(\lambda), \; Y_{n-1}(\lambda)\Big) \tag{6.1}$$

with

$$X_n(\lambda) = \begin{pmatrix} e^{-\mu\lambda} x_n \\ 1 \end{pmatrix}, \qquad Y_n(\lambda) = \begin{pmatrix} e^{-\mu\lambda} y_n \\ 1 \end{pmatrix}, \qquad x_n = x_0 e^{-i\mu n}, \quad y_n = y_0 e^{i\mu n}. \tag{6.2}$$

Here $\{x_0, y_0\} \in \mathbb{C}$ are λ−independent and will be determined explicitly in the following. Introduce also the matrices

$$M_n^{-1}(\lambda) = \begin{pmatrix} \bar{Y}_n(\lambda) \\ \bar{X}_n(\lambda) \end{pmatrix}, \quad \bar{M}_n^{-1}(\lambda) = \begin{pmatrix} \tilde{Y}_{n-1}(\lambda) \\ \tilde{X}_{n+1}(\lambda) \end{pmatrix}, \tag{6.3}$$

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with

$$\bar{X}_n(\lambda) = \frac{1}{x_n - y_n}\left(-e^{\mu\lambda}, \quad x_n\right), \quad \bar{Y}_n(\lambda) = \frac{1}{x_n - y_n}\left(e^{\mu\lambda}, \quad -y_n\right),$$

$$\tilde{X}_n(\lambda) = \frac{1}{x_n - y_{n-2}}\left(-e^{\mu\lambda}, \quad x_n\right), \quad \tilde{Y}_n(\lambda) = \frac{1}{x_{n+2} - y_n}\left(e^{\mu\lambda}, \quad -y_n\right). \tag{6.4}$$

One can check that certain combinations of (6.2), (6.4) lead to orthogonal relations among the various vectors. Also, face-vertex correspondence relations corresponding to commutation relations between the above quantities can be derived using the $R-$matrix (2.2), which are omitted here for brevity [19]. We now introduce the gauge transformed $L-$operator,

$$\overline{L}_n(m|\lambda) = \bar{M}^{-1}_{(n-1)g+m}(\lambda)L_n(\lambda)\bar{M}_{ng+m}(\lambda) \equiv \begin{pmatrix} \tilde{\alpha}_n & \tilde{\beta}_n \\ \tilde{\gamma}_n & \tilde{\delta}_n \end{pmatrix},$$

$$S_n(m|\lambda) = M^{-1}_{ng+m}(-\lambda)L_n^{-1}(-\lambda)M_{(n-1)g+m}(-\lambda) \equiv \begin{pmatrix} \tilde{\alpha}'_n & \tilde{\beta}'_n \\ \tilde{\gamma}'_n & \tilde{\delta}'_n \end{pmatrix}. \tag{6.5}$$

where g depends on the choice of representation. A priori one may associate a different representation at each site of the chain, here however we consider for the sake of simplicity the case where all sites correspond to the same representation. The occurrence of a different g at each site, depending on the choice of representation, is a non trivial observation facilitating the generalization of the formulation described [19], where all sites are associated to the fundamental representation. Note for instance, that for the spin $\frac{1}{2}$ representation the factor g is unit, for the spin s representation is 2s, and for the models we are considering here in the cyclic representation will be specified later on. Also at the end of the section 6.2 we shall discuss the special case where all sites correspond to the spin $\frac{1}{2}$ and the space attached to the boundary is associated to the cyclic representation (2.12). This scenario in fact corresponds to the Hamiltonian described in section 4 (4.8).

It follows that the transfer matrix (3.12) can be rewritten with the help of the aforementioned gauge transformations (6.5), and related orthogonal relations [19] as

$$t(\lambda) = tr_0\left\{\tilde{K}_0^+(\lambda)\ \tilde{\mathcal{T}}_0(\lambda)\right\}, \tag{6.6}$$

where

$$\tilde{K}^+(m|\lambda) = \begin{pmatrix} \tilde{K}_1^+(m|\lambda) & \tilde{K}_2^+(m|\lambda) \\ \tilde{K}_3^+(m|\lambda) & \tilde{K}_4^+(m|\lambda) \end{pmatrix} = \begin{pmatrix} \bar{Y}_m(-\lambda)K^+(\lambda)X_m(\lambda) & \bar{Y}_m(-\lambda)K^+(\lambda)Y_{m-2}(\lambda) \\ \bar{X}_m(-\lambda)K^+(\lambda)X_{m+2}(\lambda) & \bar{X}_m(-\lambda)K^+(\lambda)Y_m(\lambda) \end{pmatrix} \tag{6.7}$$

and

$$\tilde{\mathcal{T}}(\lambda) = \begin{pmatrix} \mathcal{A}_m(\lambda) & \mathcal{B}_m(\lambda) \\ \mathcal{C}_m(\lambda) & \mathcal{D}_m(\lambda) \end{pmatrix} = \begin{pmatrix} \tilde{Y}_{m-2}(\lambda)\mathcal{T}(\lambda)X_m(-\lambda) & \tilde{Y}_m(\lambda)\mathcal{T}(\lambda)Y_m(-\lambda) \\ \tilde{X}_m(\lambda)\mathcal{T}(\lambda)X_m(-\lambda) & \tilde{X}_{m+2}(\lambda)\mathcal{T}(\lambda)Y_m(-\lambda) \end{pmatrix}. \tag{6.8}$$

Similarly to [19], one defines the transformed K^- matrix as

$$\tilde{K}^-(l|\lambda) = \begin{pmatrix} \tilde{K}_1^-(l|\lambda) & \tilde{K}_2^-(l|\lambda) \\ \tilde{K}_3^-(l|\lambda) & \tilde{K}_4^-(l|\lambda) \end{pmatrix} = \begin{pmatrix} \tilde{Y}_{l-2}(\lambda)K^-(\lambda)X_l(-\lambda) & \tilde{Y}_l(\lambda)K^-(\lambda)Y_l(-\lambda) \\ \tilde{X}_l(\lambda)K^-(\lambda)X_l(-\lambda) & \tilde{X}_{m+2}(\lambda)K^-(\lambda)Y_l(-\lambda) \end{pmatrix} \tag{6.9}$$

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with $l = m + Ng$. In the case where a different representation is assigned at each site then $l = m + \sum_{n=1}^{N} g_n$, where g_n characterizes the representation at each site. Note that here the $K^{\pm}$ matrices are c-number solutions of the reflection equation, with relevant parameters $\xi^{\pm}$, $\kappa^{\pm}$, $q^{\theta^{\pm}}$.

The first step for the diagonalization of the transfer matrix (3.12) is the construction of an appropriate pseudo-vacuum. Such a state will be of the general form:

$$\Omega^{(m)} = \bigotimes_{n=1}^{N} \omega_n^{(m)}. \tag{6.10}$$

Here we denote $\omega_n^{(m)}$ the local pseudo-vacuum annihilated by the lower left elements of the transformed matrices (6.5), i.e.

$$\tilde{\gamma}_n, \ \tilde{\gamma}'_n \ \omega_n^{(m)} = 0. \tag{6.11}$$

Such a choice ensures that the operators $T(\lambda)$, $\hat{T}(\lambda)$ (3.13), acting on the pseudo-vacuum state reduce to upper triangular matrices. The above requirements (6.11) lead to certain sets of algebraic constraints. In particular, let

$$L_{0n}(\lambda) = \begin{pmatrix} \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & \delta_n(\lambda) \end{pmatrix} \tag{6.12}$$

from the action of $\tilde{\gamma}_n$ on the local pseudo-vacuum we obtain:

$$\left[-x_{m+ng+1} \, \alpha_n(\lambda) + x_{m+(n-1)g+1} \, \delta_n(\lambda) + e^{-\mu\lambda} x_{m+ng+1} \, x_{m+(n-1)g+1} \, \gamma_n(\lambda) - e^{\mu\lambda}\beta_n(\lambda) \right] \omega_n^{(m)} = 0 \tag{6.13}$$

from the action of $\tilde{\gamma}'_n$ a similar constraint is entailed.

The non-diagonal elements of (6.7), (6.9) acting on the pseudo-vacuum state (6.10) must satisfy certain constraints such that the pseudo-vacuum is indeed an eigenstate of the transfer matrix (6.6). Indeed, we associate the integers m^0 and m_0 to the left and right boundaries respectively, and impose the following:

$$\tilde{K}_2^+(m^0|\lambda) = \tilde{K}_3^+(m^0|\lambda) = 0 , \quad \tilde{K}_3^-(m_0|\lambda) = 0 , \tag{6.14}$$

$\tilde{K}_{2,3}^{\pm}$ are given in (6.7), (6.9). The above constraints may be solved [19] fixing the relations among the parameters m_0, m^0, x_0, y_0. Indeed let $x_0 = -ie^{-i\mu(\beta+\gamma)}$, $y_0 = -ie^{-i\mu(\beta-\gamma)}$, then the relations among the boundary parameters (6.14) are

$$\frac{e^{-i\mu\xi^+}}{2\kappa^+} = -i\cosh i\mu(\beta^+ + \gamma^+ + 1), \quad \frac{e^{-i\mu\xi^+}}{2\kappa^+} = -i\cosh i\mu(\beta^+ - \gamma^+ + 1),$$

$$\frac{e^{-i\mu\xi^-}}{2\kappa^-} = i\cosh i\mu(\beta^- + \gamma^-), \tag{6.15}$$

and we define

$$\gamma^+ = \gamma + m^0, \quad \beta^+ = \beta + \theta^+, \quad \gamma^- = \gamma + m_0, \quad \beta^- = \beta + \theta^-. \tag{6.16}$$

For relevant discussions on these constraints see also [20, 18, 21, 19]. It would be illuminating to consider the above constraints bearing in mind the identifications of the boundary parameter with the parameters

of the blob algebra [58]. Indeed analogous relations emerge in the double Temperley–Lieb algebra for exceptional values of the algebra parameters in [23]. It is not clear to us however why such constraints hold true for generic values of the boundary parameters. The natural question raised is whether this is simply a disadvantage of the methods employed or it reflects a deeper physical or algebraic reason. This is an intricate point, which merits further investigation.

In what follows we examine the pseudo-vacua for both the XXZ model, and the q harmonic oscillator. Their explicit forms are specified by solving the formulas (6.13), which lead to sets of difference equations.

(**I**) *The XXZ model*: As already mentioned we shall treat the cyclic representation (2.5), (2.10), (2.12), for the special case where q is root of unity. We associate each site of the chain with the cyclic representation, and we define the local pseudo-vacuum state as

$$\omega_n^{(m)} = \sum_{i=1}^{p} w_l^{(m,n)} \, f_l^{(n)} \tag{6.17}$$

$f_l^{(n)}$ is a p dimensional column vector with zeroes everywhere apart from the lth position. The constraints (6.13) provide (a) the value of $\mathbf{g = 2s - 2}$ and (b) recursion relations among the $w_k^{(m,n)}$:

$$\frac{w_{k+1}^{(m,n)}}{w_k^{(m,n)}} = \frac{q^{s-1+\Theta} \sinh i\mu(k - s + 1)}{x_{m+(n-1)g+1} \, \sinh i\mu(k + s)}, \qquad \frac{w_p^{(m,n)}}{w_1^{(m,n)}} = x_{m+(n-1)g+1} \, q^{-\Theta+1-s} \frac{\sinh(i\mu s)}{\sinh i\mu(1 - s)}. \tag{6.18}$$

By normalizing the state with $w_1^{(m,n)} = 1$ we can write

$$w_k^{(m,n)} = \Big(\frac{q^{s-1+\Theta}}{x_{m+(n-1)g+1}} \Big)^{k-1} \prod_{j=1}^{k-1} \frac{\sinh i\mu(j - s + 1)}{\sinh i\mu(j + s)}. \tag{6.19}$$

Comparing the expressions of $w_p^{(m,n)}$ from equations (6.18), (6.19) we obtain the following constraint on x_m, i.e.

$$\Big(- x_{m+(n-1)g+1} \, q^{-\Theta+1-s} \Big)^p = 1. \tag{6.20}$$

Relations (6.18) are sufficient for the derivation of the spectrum.

(**II**) *The q harmonic oscillator*: For the L matrix describing the q harmonic oscillator (2.24) we consider $X \to q^z X$, $Y \to q^{\frac{1}{2}} Y$ (cyclic representation (2.12)). In this case $\mathbf{g = -1}$. The local pseudo-vacuum will be also of the form (6.17), and the corresponding recursion relations are also provided by solving the constraints (6.13),

$$\frac{w_{k+1}^{(m,n)}}{w_k^{(m,n)}} = -\frac{2 \sinh i\mu(k - z + \frac{1}{2})}{x_{m+ng+1} \, q^{-k+z-1-\Theta}}, \qquad \frac{w_p^{(m,n)}}{w_1^{(m,n)}} = -\frac{x_{m+ng+1} \, q^{z-1-\Theta}}{2 \sinh i\mu(\frac{1}{2} - z)}. \tag{6.21}$$

Again normalizing the pseudo-vacuum with $w_1^{(m,n)} = 1$ we conclude that

$$w_k^{(m,n)} = \Big(-\frac{2q^{-z+1+\Theta}}{x_{m+ng+1}} \Big)^{k-1} \prod_{j=1}^{k-1} q^j \sinh i\mu(j - z + \frac{1}{2}), \tag{6.22}$$

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and the constraint necessary from cyclicity leads to:

$$\left(-\frac{1}{2}x_{m+1+ng}\ q^{z-1-\Theta}\right)^p = \prod_{k=1}^{p} q^k \sinh i\mu(k + \frac{1}{2} - z).$$

(6.23)

To derive the spectrum of the transfer matrix for both the XXZ and the q harmonic oscillator we also need to know the action of the transformed diagonal elements on the pseudo-vacuum, which are given by:

$$\tilde{\alpha}_n^{(m)}\ \omega_n^{(m)} = a(\lambda)\ \omega_n^{(m+1)},$$

$$\tilde{\delta}_n^{(m)}\ \omega_n^{(m)} = \frac{x_{m+ng+1} - y_{m+ng-1}}{x_{m+(n-1)g+1} - y_{m+(n-1)g-1}}\ b(\lambda)\ \omega_n^{(m-1)}$$

$$\tilde{\alpha}_n'^{(m)}\ \omega_n^{(m)} = a'(\lambda)\ \omega_n^{(m-1)},$$

$$\tilde{\delta}_n^{('m)}\ \omega_n^{(m)} = \frac{x_{m+(n-1)g} - y_{m+(n-1)g}}{x_{m+ng} - y_{m+ng}}\ b'(\lambda)\ \omega_n^{(m+1)}.$$

(6.24)

The values of $a,\ b,\ a',\ b'$ for each representation are given below

$$(\mathbf{I}) \quad a(\lambda) = q^{-s+1}\ \sinh\mu(\lambda + is - i\Theta - i), \qquad b(\lambda) = q^{s-1}\ \sinh\mu(\lambda - is - i\Theta + i)$$

$$a'(\lambda) = q^{-1+s}\ \sinh\mu(\lambda + is + i\Theta), \qquad b'(\lambda) = q^{1-s}\ \sinh\mu(\lambda - is + i\Theta + 2i)$$

$$(\mathbf{II}) \quad a(\lambda) = \frac{1}{2}q^{1-z+\Theta}e^{-\mu\lambda}, \qquad b(\lambda) = q^{z-\frac{1}{2}}\ \sinh\mu(\lambda - i\Theta + \frac{i}{2})$$

$$a'(\lambda) = q^{-\frac{1}{2}+z}\sinh\mu(\lambda + \frac{i}{2} + i\Theta), \qquad b'(\lambda) = \frac{1}{2}q^{2-z}\ e^{\mu(\lambda+i\Theta)}.$$

(6.25)

6.2   Spectrum and Bethe ansatz equations

Now that the appropriate gauge transformations and the corresponding pseudo-vacua have been explicitly derived, we can proceed with the computation of the eigenvalues of the transfer matrix. Namely, we are looking for the solution of the following eigenvalue problem

$$t(\lambda)\ \Psi = \Lambda(\lambda)\ \Psi$$

(6.26)

where Ψ is the general Bethe ansatz state of the form

$$\Psi = \mathcal{B}_{m^0-2}(\lambda_1)\ldots\mathcal{B}_{m^0-2M}(\lambda_M)\ \Omega^{(m)}\ ,$$

(6.27)

and m will be defined below [7, 19]. Then one has to solve a typical eigenvalue problem (6.26) written explicitly as

$$t(\lambda)\ \Psi = \left(K_1^+(m^0|\lambda)\mathcal{A}_{m^0}(\lambda) + K_4^+(m^0|\lambda)\tilde{\mathcal{D}}_{m^0}(\lambda)\right)\ \Psi,$$

(6.28)

where

$$\tilde{\mathcal{D}}_m(\lambda) = \frac{\sinh\mu(2\lambda + i)\sinh\mu(im + i\gamma + i)}{\sinh(i\mu)\sinh i\mu(m + \gamma)}\mathcal{D}_m(\lambda) - \frac{\sinh\mu(im + i\gamma + 2\lambda + i)}{\sinh i\mu(m + \gamma)}\mathcal{A}_m(\lambda)$$

and the elements $K_{1,4}^+(m^0|\lambda)$ are given in Appendix B. Using the definition of $\mathcal{T}(\lambda)$ in (3.12) and (6.8), the action of the operators $\mathcal{A}_m(\lambda)$ and $\tilde{\mathcal{D}}_m(\lambda)$ on the bulk part of the pseudo-vacuum (6.10) may be

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derived. Assuming that each site n of the chain is associated to either the XXZ model (2.5) or to the q harmonic oscillator (2.24) in the cyclic representation (2.12), it is convenient to define

$$f_n(\lambda) = a(\lambda)\ a'(\lambda), \qquad h_n(\lambda) = b(\lambda)\ b'(\lambda), \tag{6.29}$$

a, b, a', b' are defined in (6.25). Then bearing in mind relations (6.24) and appropriate algebraic relations arising from the reflection equation (see also [19]) we obtain:

$$
\begin{aligned}
\mathcal{A}_m(\lambda)\ \Omega^{(m)} &= \prod_{n=1}^{N} f_n(\lambda)\ K_1^-(m_0|\lambda)\ \Omega^{(m)} \\
\tilde{\mathcal{D}}_m(\lambda)\ \Omega^{(m)} &= \prod_{n=1}^{N} h_n(\lambda)\ K_4^-(m_0|\lambda)\ \Omega^{(m)},
\end{aligned}
\tag{6.30}
$$

$m_0 = m + gN$, and $K_{1,4}^-(m_0|\lambda)$ are introduced in Appendix A.

Having in mind the structure of the general Bethe state (6.27) it is obvious that we need exchange relations between the operators $\mathcal{A}_m(\lambda)$ and $\tilde{\mathcal{D}}_m(\lambda)$ with $\mathcal{B}_l(\lambda)$, which can be deduced with the help of the reflection equation (3.1) (see also [19]), then one obtains

$$
\begin{aligned}
\mathcal{A}_{m+2}(\lambda_1)\mathcal{B}_m(\lambda_2) =\ & \frac{\sinh\mu(\lambda_1 + \lambda_2)\ \sinh\mu(\lambda_1 - \lambda_2 - i)}{\sinh\mu(\lambda_1 - \lambda_2)\ \sinh\mu(\lambda_1 + \lambda_2 + i)}\mathcal{B}_m(\lambda_2)\mathcal{A}_m(\lambda_1) \\
& -\frac{\sinh(2\mu\lambda_2)\ \sinh(i\mu)\ \sinh\mu(\lambda_1 - \lambda_2 - im - i\gamma - i)}{\sinh\mu(im + i\gamma + i)\ \sinh\mu(\lambda_1 - \lambda_2)\ \sinh\mu(2\lambda_2 + i)}\mathcal{B}_m(\lambda_1)\mathcal{A}_m(\lambda_2) \\
& -\frac{\sinh(i\mu)\ \sinh(i\mu)\ \sinh\mu(-\lambda_1 - \lambda_2 + im + i\gamma)}{\sinh\mu(im + i\gamma + i)\ \sinh\mu(\lambda_1 + \lambda_2 + i)\ \sinh\mu(2\lambda_2 + i)}\mathcal{B}_m(\lambda_1)\tilde{\mathcal{D}}_m(\lambda_2)
\end{aligned}
\tag{6.31}
$$

and

$$
\begin{aligned}
\tilde{\mathcal{D}}_{m+2}(\lambda_1)\mathcal{B}_m(\lambda_2) =\ & \frac{\sinh\mu(\lambda_1 + \lambda_2 + 2i)\ \sinh\mu(\lambda_1 - \lambda_2 + i)}{\sinh\mu(\lambda_1 - \lambda_2)\ \sinh\mu(\lambda_1 + \lambda_2 + i)}\mathcal{B}_m(\lambda_2)\tilde{\mathcal{D}}_m(\lambda_1) \\
& -\frac{\sinh\mu(2\lambda_1 + 2i)\ \sinh(i\mu)\ \sinh\mu(\lambda_1 - \lambda_2 + im + i\gamma + i)}{\sinh\mu(im + i\gamma + i)\ \sinh\mu(\lambda_1 - \lambda_2)\ \sinh\mu(2\lambda_2 + i)}\mathcal{B}_m(\lambda_1)\tilde{\mathcal{D}}_m(\lambda_2) \\
& +\frac{\sinh(2\mu\lambda_2)\ \sinh 2\mu(\lambda_1 + i)\ \sinh\mu(\lambda_1 + \lambda_2 + im + i\gamma + 2i)}{\sinh\mu(im + i\gamma + i)\ \sinh\mu(\lambda_1 + \lambda_2 + i)\ \sinh\mu(2\lambda_2 + i)}\mathcal{B}_m(\lambda_1)\mathcal{A}_m(\lambda_2)\ .
\end{aligned}
\tag{6.32}
$$

In fact all the above exchange relations, arising from the reflection equation, hold exactly as described in [19]. What is only modified is the action of the diagonal elements and consequently of $\mathcal{A}_m$, $\tilde{\mathcal{D}}_m$ on the pseudo-vacuum (6.30), which clearly depends on the choice of the particular representation. It may be shown that the state Ψ is indeed an eigenstate of the transfer matrix if we impose $m \equiv m^0 - 2M$. Without loosing generality we assume for simplicity $m^0 = 0$, and then it immediately follows that

$$m_0 = Ng - 2M\ . \tag{6.33}$$

We may now deduce the transfer matrix eigenvalues by virtue of equations (6.28), (6.30), (6.31), (6.32) i.e.

$$
\begin{aligned}
\Lambda(\lambda) = \Big(& K_1^+(0|\lambda)K_1^-(m_0|\lambda) \prod_{n=1}^{N} f_n(\lambda)\ \prod_{j=1}^{M} \frac{\sinh\mu(\lambda + \lambda_j)\ \sinh\mu(\lambda - \lambda_j - i)}{\sinh\mu(\lambda + \lambda_j + i)\ \sinh\mu(\lambda - \lambda_j)} \\
+\ & K_4^+(0|\lambda)K_4^-(m_0|\lambda) \prod_{n=1}^{N} h_n(\lambda)\ \prod_{j=1}^{M} \frac{\sinh\mu(\lambda + \lambda_j + 2i)\ \sinh\mu(\lambda - \lambda_j + i)}{\sinh\mu(\lambda + \lambda_j + i)\ \sinh\mu(\lambda - \lambda_j)}\Big)
\end{aligned}
\tag{6.34}
$$

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provided a certain combination of 'unwanted' terms arising from the commutation relations (6.31), (6.32) that appear in the eigenvalue expression, is vanishing. This is true as long as λ_i's satisfy a set of conditions, namely the Bethe ansatz equations. These equations guarantee also analyticity of the eigenvalues, and they are written in the familiar form

$$H(\lambda_i) \prod_{n=1}^{N} \frac{f_n(\lambda_i - \frac{i}{2})}{h_n(\lambda_i - \frac{i}{2})} = -\prod_{j=1}^{M} e_2(\lambda_i - \lambda_j) \, e_2(\lambda_i + \lambda_j) \, , \tag{6.35}$$

where we define

$$e_n(\lambda) = \frac{\sinh \mu(\lambda + \frac{in}{2})}{\sinh \mu(\lambda - \frac{in}{2})} \quad \text{and} \quad H(\lambda) = \frac{K_1^+(0|\lambda - \frac{i}{2}) \, K_1^-(m_0|\lambda - \frac{i}{2})}{K_4^+(0|\lambda - \frac{i}{2}) \, K_4^-(m_0|\lambda - \frac{i}{2})} \, . \tag{6.36}$$

A few comments are in order at this point. Notice that for the q harmonic oscillator a factor q^{-N}, which reduces to ± 1 for $N = kp$ (k integer or half integer), occurs in the left hand side of the Bethe ansatz equations, resembling the case of the XXZ chain with twisted boundary conditions. As realized in [41] a similar factor appears also in the Bethe ansatz equations of the lattice Liouville model with periodic boundary conditions. Furthermore, for particular values of the parameter s and the inhomogeneities Θ i.e. (I) $s = 1$, (II) $\Theta = 0$ (see also (6.25)), the Bethe ansatz equations become degenerate, and as a consequence the Bethe ansatz formulation is not appropriate any more for deriving the spectrum. In this case one has to deal with the relevant Baxter operators and the Baxter equations [4] in order to examine the associated spectra.

The case of dynamical boundaries studied in a series of papers (see e.g. [7, 13], [26]–[24] and references therein) can be regarded as a special case of the above description. Indeed, let us consider a typical example of dynamical boundaries, which is described by the Hamiltonian of (4.8). Consider a chain with $N + 1$ sites, and assign to all quantum spaces $n \in \{1, \ldots, N\}$ the spin $\frac{1}{2}$ representation, whereas the $N + 1$ site is associated to the cyclic representation, of $\mathcal{U}_q(sl_2)$ or the q harmonic oscillator algebra that is we consider a dynamical boundary at the right end (see also section 4). In fact, the $N + 1$ space is the 'dynamical' space d of section 4. Then the spectrum takes the form (6.34), but with the functions f_n, h_n being (set here for simplicity all the inhomogeneities to zero):

$$f_n(\lambda - \frac{i}{2}) = \sinh^2 \mu(\lambda + \frac{i}{2}), \quad h_n(\lambda - \frac{i}{2}) = \sinh^2 \mu(\lambda - \frac{i}{2}), \quad n \in \{1, \ldots, N\}$$

$$f_{N+1}(\lambda - \frac{i}{2}) = a(\lambda - \frac{i}{2}) \, a'(\lambda - \frac{i}{2}), \quad h_{N+1}(\lambda - \frac{i}{2}) = b(\lambda - \frac{i}{2}) \, b'(\lambda - \frac{i}{2}), \tag{6.37}$$

and recall $a(\lambda)$, $a'(\lambda)$, $b(\lambda)$, $b'(\lambda)$ are given in (6.25) for both $\mathcal{U}_q(sl_2)$ and the q harmonic oscillator. The pseudovacuum for the spin $\frac{1}{2}$ case have been found in [19] so we do not repeat them here. The local pseodovacuum for the site n is given by (6.17) and (6.19) or (6.22) depending on the dynamics at the boundary i.e. $\mathcal{U}_q(sl_2)$ or q harmonic oscillator respectively. The general Bethe states are given in (6.27). We could have considered a dynamical boundary in the left end as well, then $f_1 = f_{N+1}$, $h_1 = h_{N+1}$.

6.3   Asymptotics and derivation of M

In [18, 19] the spin $\frac{1}{2}$ XXZ model with both boundaries being non-diagonal is discussed in detail. In this case via the asymptotic behavior of the transfer matrix it was shown that the value of the integer M is

fixed (see also [20, 21] for the spin s representation). Let us now consider a special left boundary given by (5.15), then the value of M may be derived in terms of the spectrum of the conserved quantity $Q_1^{(N)}$. Indeed the asymptotic expansion of the transfer matrix (6.34) as $\mu\lambda \to \infty$, taking also into account (B.6), is given for (I) the XXZ model, and (II) the q harmonic oscillator:

(I) $\quad \Lambda(\mu\lambda \to \infty) = -i \; \kappa^- \; e^{2\mu\lambda(N+1)+i\mu(N+1)} \left(e^{i\mu(\beta^-+\gamma^--2(s-1)N+2M)} + e^{-i\mu(\beta^-+\gamma^--2(s-1)N+2M)} \right)$

(II) $\quad \Lambda(\mu\lambda \to \infty) = -i \; \kappa^- \; e^{2\mu\lambda(N+1)+i\mu(2N+1)} e^{i\mu(\beta^-+\gamma^-+2M)}$ $\hfill (6.38)$

comparing with the results of section 5, and taking into account appropriate normalizations for the K and R matrices, we conclude,

(I) $\quad Q_1^{(N)} = i \; \sinh^{-1}(i\mu) \left(q^{\beta^-+\gamma^-+2M-2(s-1)N} + q^{-\beta^--\gamma^--2M+2(s-1)N} \right)$

(II) $\quad Q_1^{(N)} = 2 \; i \; q^{\beta^-+\gamma^-+2M+N}. \hfill (6.39)$

The expressions of the boundary charges for both XXZ and q harmonic oscillator are given by the expressions (5.5) and (5.14) respectively. Also the sum $\beta^- + \gamma^-$ is related to the right boundary parameters of the chain (6.15). From equation (6.39) the upper (lower) bounds of M (integer) are identified by means of the spectrum of the non-local operator $Q_1^{(N)}$, which plays the role of q^{s^z}, the corresponding conserved quantity when diagonal boundaries are applied in the spin chain. The relation between $Q_1^{(N)}$ and M underlines the significance of the algebraic object $Q_1^{(N)}$ into the derivation of the spectrum of the open spin chain with special boundary conditions. The identification of the spectrum of the boundary non-local charge is an intriguing problem, and some progress has been already achieved towards this direction, for particular representations in [23], however the problem in its full generality remains open. In fact, the operator $Q_1^{(N)}$ locally consists of tridiagonal form matrices for the spin s representation (2.7) (we provide an example in Appendix C). The explicit diagonalization of $Q_1^{(N)}$ for generic number of sites N will be discussed elsewhere.

Similar algebraic objects, forming the corresponding boundary quantum algebra [61], are necessary for the identification of the spectrum of models associated to higher rank algebras with non diagonal right boundary.

7   Boundary Lattice sine–Gordon and Liouville models

In this section we give a flavor on the lattice sine–Gordon and Liouville theories with diagonal boundaries, that is in the expressions for the c-number $K^\pm$ matrices we set $\kappa^\pm = 0$. In particular, we identify the spectrum and the corresponding Bethe ansatz equations for both models.

In what follows we consider for simplicity the inhomogeneity Θ to be zero. We find local pseudo-vacua annihilating the lower left element of $\mathbb{L}$, $\hat{\mathbb{L}}$, being effectively 'highest weight' states. Let such a state be of the form [40, 41]

$$\omega_n = f(\Phi_{2n}) \; \delta(\Phi_{2n} - \Phi_{2n-1} - \mu) \hfill (7.1)$$

$$\gamma_n, \; \hat{\gamma}_n \; \omega_n = 0 \hfill (7.2)$$

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and consequently $f(\Phi)$ should be a solution of the following set of difference equations

$$\text{Lattice sine–Gordon:} \qquad f(\Phi + \mu) = -\frac{h_-(\Phi)}{h_+(\Phi)} f(\Phi)$$

$$\text{Lattice Liouville:} \qquad f(\Phi + \mu) = -h(\Phi) f(\Phi) \tag{7.3}$$

which occur exactly as in the bulk case. When q is root of unity a cyclicity (periodicity) requirement is necessary to be imposed on the pseudo-vacuum as well, i.e. (see also [40, 41]),

$$f(\Phi + p\mu) = f(\Phi). \tag{7.4}$$

To find the spectrum of the lattice systems we also need the explicit action of the diagonal entries of the $\mathbb{L}$, $\hat{\mathbb{L}}$ matrices on the local pseudo-vacuum. Indeed it is straightforward to show that

$$\alpha_n(\lambda) \, \omega_n = a(\lambda) \, \omega_n, \qquad \hat{\alpha}_n(\lambda) \, \omega_n = a'(\lambda) \, \omega_n,$$
$$\delta_n(\lambda) \, \omega_n = b(\lambda) \, \omega_n, \qquad \hat{\delta}_n(\lambda) \, \omega_n = b'(\lambda) \, \omega_n \tag{7.5}$$

where

$$\text{Lattice sine–Gordon:} \qquad a(\lambda) = -4m^2 \cosh \mu(\lambda - \frac{i}{2} + \frac{ir}{2}) \, \cosh \mu(\lambda - \frac{i}{2} - \frac{ir}{2})$$

$$b(\lambda) = -4m^2 \cosh \mu(\lambda + \frac{i}{2} + \frac{ir}{2}) \, \cosh \mu(\lambda + \frac{i}{2} - \frac{ir}{2})$$

$$a'(\lambda) = -4m^2 \cosh \mu(\lambda + \frac{i}{2} + \frac{ir}{2}) \, \cosh \mu(\lambda + \frac{i}{2} - \frac{ir}{2})$$

$$b'(\lambda) = -4m^2 \cosh \mu(\lambda + \frac{3i}{2} + \frac{ir}{2}) \, \cosh \mu(\lambda + \frac{3i}{2} - \frac{ir}{2})$$

$$\text{Lattice Liouville:} \qquad a(\lambda) = -e^{-\mu(\lambda - \frac{i}{2})} \, \sinh \mu(\lambda - \frac{i}{2})$$

$$b(\lambda) = -e^{-\mu(\lambda + \frac{i}{2})} \, \sinh \mu(\lambda + \frac{i}{2})$$

$$a'(\lambda) = e^{\mu(\lambda + \frac{i}{2})} \, \sinh \mu(\lambda + \frac{i}{2})$$

$$b'(\lambda) = e^{\mu(\lambda + \frac{3i}{2})} \, \sinh \mu(\lambda + \frac{3i}{2}), \tag{7.6}$$

the constant r appearing in the sine–Gordon eigenvalues is given by $2 \cosh i\mu r = (m^2 + m^{-2})$, it plays the role of inhomogeneity and it is associated to the mass scale of the system. The standard algebraic Bethe ansatz may be applied [7], and the spectrum is obtained having the familiar form of (6.34) $(K_{1,4}^-(m_0|\lambda) \to K_{1,4}^-(\xi^-|\lambda), \; K_{1,4}^+(0|\lambda) \to K_{1,4}^+(\xi^+|\lambda))$, while the corresponding Bethe ansazt equations as customary follow as analyticity conditions on the spectrum and they have the form (6.35), (6.36) with

$$f_n(\lambda) = a(\lambda) \, a'(\lambda), \qquad h_n(\lambda) = b(\lambda) \, b'(\lambda)$$

$$K_1^-(\xi^-|\lambda) \, K_1^+(\xi^+|\lambda) = \frac{\sinh \mu(2\lambda + 2i)}{\sinh \mu(2\lambda + i)} \sinh \mu(-\lambda + i\xi^-) \, \sinh \mu(\lambda + i\xi^+)$$

$$K_4^-(\xi^-|\lambda) \, K_4^+(\xi^+|\lambda) = \frac{\sinh(2\mu\lambda)}{\sinh \mu(2\lambda + i)} \sinh \mu(\lambda + i\xi^- + i) \, \sinh \mu(-\lambda + i\xi^+ - i), \tag{7.7}$$

and $a(\lambda)$, $a'(\lambda)$, $b(\lambda)$, $b'(\lambda)$ given by (7.6) for both the lattice sine-Gordon and Liouville models. The integer M appearing in the Bethe ansatz (6.35) this time is associated to the spectrum of the topological charge $K_1^{(N)}$ (5.6), (5.9) for the sine-Gordon model. It should be stressed that the identification of the boundary non-local charges for the case of the open lattice Liouville theory is an intriguing task, which needs to be further pursued.

In the same spirit the case where the right boundary is upper (lower) triangular may be examined. An obvious reference state still exists, and the Bethe ansatz equations may be easily extracted being of the familiar form. In this case of course the corresponding conserved quantity for the lattice sine-Gordon will be the charge $Q_1^{(N)}$ (5.5) (for a trivial left boundary (5.15)), but with the terms proportional to $q^{-\theta^-}$ (q^{θ^-}) omitted. Finally the more general scenario with non-diagonal boundaries may be studied employing appropriate local gauge transformations. The spectrum is anticipated to be of the form (6.34), but this time the integer M should be associated to the corresponding conserved quantity. In general M being associated to the corresponding conserved non-local charge reflects essentially the underlying symmetry. Moreover, in this case the derivation of the reference state involves the solution of more complicated sets of difference equations, and it will be left for future investigations.

8 Conclusions

The main aim of the present study was the investigation of a class of open quantum integrable models emerging from the generalized XXZ model (2.5). In this spirit a particular example of a spin $\frac{1}{2}$ XXZ Hamiltonian coupled to a dynamical boundary (4.8) was first considered. Explicit expressions of the corresponding dynamical boundary term associated to the two main algebras i.e the $\mathcal{U}_q(sl_2)$ and the q harmonic oscillator algebra were derived (4.9), (4.10). The main point is that the expressions of the boundary terms (4.9) and (4.10) are generic at this stage that is independent of the choice of representation, hence they can be exploited for a variety of dynamical systems (quantum impurities) coupled at the end if the chain. As a step towards the identification of the spectrum of the various models under consideration, and after recalling known expressions of the boundary non-local charges for the generalized XXZ model, we derived novel expressions of boundary non-local charges for the lattice sine Gordon model (5.5) and (5.9), and for the q harmonic oscillator (5.13) and (5.14). The significance of such quantities and their relevance to the spectrum is emphasized in section 6.3.

In order to examine the spectra and eigenstates of the XXZ model and the q harmonic oscillator with non diagonal boundaries in the cyclic representation (2.12), we generalized the approach presented in [19]. The crucial observation is that although the local gauge transformations (see (6.5)) act on the 'auxiliary' space —being always associated to the fundamental representation of $\mathcal{U}_q(sl_2)$— they depend explicitly on the choice of representation for each quantum space (site). In particular, there exists a parameter g incorporated in the local transformations (6.5), whose value depends clearly on the choice of representation for the each site (for the spin $\frac{1}{2}$ case is unit). Based on this observation we were able to derive explicit expressions for the pseudovacua (6.17), (6.19), (6.20), (6.22), (6.23) by solving sets of involved recursion relations (6.18), (6.21). In fact both the value of g and the recursion relations (6.18), (6.21) were deduced from the solution of the constraint (6.13), which is the fundamental object in this construction. More precisely, the main requirement imposed was that the transformed L matrices

(6.5) become upper triangular after acting on the suitable local pseudo vacuum, which eventually led to the constraint (6.13). Consequently the Bethe states (6.27) were identified, and the spectra (6.25), (6.34), (6.29) and Bethe ansatz equations (6.35) of the aforementioned models were derived. The derivation of the pseodovacua and Bethe states is of particular interest especially in the context of computing correlations functions (see e.g. [14]). Moreover in the semiclassical limit as known Gaudin type Hamiltonians arise, whose eigenstates are the semiclassical limits of the Bethe states, and they satisfy Knizhnik-Zamolodchikov equations (see e.g [62]). It is a compelling task to identify the exact type of the Knizhnik-Zamolodchikov equations for the semiclassical limit of the models investigated here, this however will be undertaken elsewhere. We should also point out that within the spirit described above the Hamiltonian (4.8), with a fixed representation (cyclic) at the right end, was treated as a special case and the corresponding spectrum was presented in (6.34), (6.37).

Another intriguing new result is the link between the spectrum of the open transfer matrix with a trivial left boundary and a general non diagonal right boundary, and the conserved quantity $Q_1^{(N)}$ (see (6.38), (6.39)), which plays essentially the role that S^z plays in models with both diagonal boundaries. The study of its spectrum was also emphasized and examples on the diagonalization of the one site charge for the spin s and the cyclic representation (2.12) of the $\mathcal{U}_q(sl_2)$ were also presented (see Appendix C). Finally, we were able to identify the spectrum (6.34) and (7.7) and the corresponding Bethe ansatz equations (6.35), (7.7) for the lattice versions of the sine Gordon and Liouville models, but only with diagonal boundaries. We also realized that the corresponding pseudovacua (7.1), (7.3), (7.4) have the same structure as in the periodic case [40, 41]. Note that this is the first time that these models with open boundaries are examined, so the aforementioned results are quite useful especially from the point of view of computing the corresponding exact boundary S matrices and boundary thermodynamic properties.

It is worth emphasizing that all the models under consideration share a common spectrum form (6.34). This is somehow anticipated given that they all belong to the same universality class, emerging from the generalized XXZ model (2.5). More precisely, the terms $\prod_{i=1}^{M}$ in (6.34), and consequently the right hand side of the Bethe asnatz equations (6.35), are generic that is independent of the choice of representation. On the other hand the terms of the spectrum including f_n, h_n and the left hand side of the Bethe ansatz equations depend clearly on the choice of representation, hence the various models naturally give rise to distinct expressions. Indeed notice the various f_n, h_n functions for each model (6.25), (6.29), (6.37), (7.6), (7.7). Furthermore, the spectrum (6.34), and the left hand side of the Bethe ansatz equations (6.35), depend on the choice of boundaries see e.g. $K_{1,4}^{\pm}$ in (6.34), and also the function $H(\lambda)$ (6.36), (7.7) associated to the corresponding boundary conditions. It should be finally stressed that in order to obtain 'numbers' for the spectrum one has to solve the corresponding Bethe ansatz equations. This can be achieved numerically for finite size chains, or using thermodynamic techniques as $N \to \infty$, obtaining consequential (boundary) scattering information and (boundary) thermodynamic quantities (see e.g. [20]). This is a particularly interesting aspect, but it is beyond the intended scope of the present study, and it will be pursued in a forthcoming work.

A   The quantum Kac–Moody algebra $\mathcal{U}_q(\widehat{sl_2})$

We briefly review some basic definitions concerning the quantum group structures. Let

$$(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \tag{A.1}$$

be the Cartan matrix of the affine Lie algebra $\widehat{sl_2}$ [63], and also define

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{A.2}$$

Recall that the quantum affine enveloping algebra $\mathcal{U}_q(\widehat{sl_2}) \equiv \mathcal{A}$ has the Chevalley-Serre generators [53, 64] e_i, f_i, k_i, $i \in \{1,\ 2\}$ obeying the defining relations

$$k_i\ k_j = k_j\ k_i, \quad k_i\ e_j = q^{\frac{1}{2}a_{ij}}e_j\ k_i, \quad k_i\ f_j = q^{-\frac{1}{2}a_{ij}}f_j\ k_i,$$
$$\left[e_i,\ f_j\right] = \delta_{ij}\frac{k_i^2 - k_i^{-2}}{q - q^{-1}}, \quad i,j \in \{1,\ 2\} \tag{A.3}$$

and the q deformed Serre relations

$$\chi_i^3\ \chi_j - [3]_q\ \chi_i^2\ \chi_j\ \chi_i + [3]_q\ \chi_i\ \chi_j\ \chi_i^2 - \chi_j\ \chi_i^3 = 0, \quad \chi_i \in \{e_i,\ f_i\}, \quad i \neq j. \tag{A.4}$$

There exists a homomorphism called the evaluation homomorphism [53] $\pi_\lambda : \mathcal{U}_q(\widehat{sl_2}) \to \mathcal{U}_q(sl_2)$

$$\pi_\lambda(e_1) = e_1, \quad \pi_\lambda(f_1) = f_1, \quad \pi_\lambda(k_1) = k_1$$
$$\pi_\lambda(e_2) = e^{-2\mu\lambda}cf_1, \quad \pi_\lambda(f_2) = e^{2\mu\lambda}c^{-1}e_1, \quad \pi_\lambda(k_2) = k_1^{-1}, \tag{A.5}$$

c is a constant. As mentioned this algebra is also equipped with a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, in particular the generators form the following coproducts

$$\Delta(y) = k_i \otimes y + y \otimes k_i^{-1}, \quad y \in \{e_i,\ f_i\} \quad \text{and} \quad \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}. \tag{A.6}$$

The L-fold coproduct is derived using the recursion relations

$$\Delta^{(L)} = (\mathrm{id} \otimes \Delta^{(L-1)})\Delta \tag{A.7}$$

and thus one may obtain explicit expressions of the L coproduct as

$$\Delta^{(L)}(y) = \sum_{n=1}^{L} k_i \otimes \ldots \otimes k_i \otimes \underbrace{y}_{n \text{ position}} \otimes k_i^{-1} \otimes \ldots \otimes k_i^{-1} \quad y \in \{e_i,\ f_i\}, \quad \Delta^{(L)}(k_i) = \bigotimes_{n=1}^{L} k_{i,n}. \text{ (A.8)}$$

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B   Transformed K matrices

Explicit expressions for diagonal elements of the transformed K matrices are provided below

$$\tilde{K}_1^+(m^0|\lambda) = -\frac{e^{-\mu\lambda}\kappa^+}{\sinh(i\mu\gamma^+)}\left(\frac{e^{i\mu\xi^+}}{2\kappa^+}\sinh i\mu(1-\gamma^+) + \frac{e^{-i\mu\xi^+}}{2\kappa^+}\sinh\mu(2\lambda+i\gamma^++i) + 2i\cosh(i\mu\beta^+)\sinh(2\mu\lambda)\right)$$

$$\tilde{K}_4^+(m^0|\lambda) = \frac{e^{-\mu\lambda}\kappa^+}{\sinh(i\mu\gamma^+)}\left(\frac{e^{i\mu\xi^+}}{2\kappa^+}\sinh i\mu(1+\gamma^+) + \frac{e^{-i\mu\xi^+}}{2\kappa^+}\sinh\mu(2\lambda-i\gamma^++i) + 2i\cosh(i\mu\beta^+)\sinh(2\mu\lambda)\right)$$

$$\text{(B.1)}$$

$$
\begin{aligned}
\tilde{K}_1^-(m_0|\lambda) &= \frac{e^{\mu\lambda}\kappa^-}{\sinh i\mu(\gamma^--1)}\left(\frac{e^{i\mu\xi^-}}{2\kappa^-}\sinh i\mu(-1+\gamma^-) + \frac{e^{-i\mu\xi^-}}{2\kappa^-}\sinh\mu(2\lambda-i\gamma^-+i)\right. \\
&\quad - \left. 2i\cosh i\mu(\beta^-+1)\sinh(2\mu\lambda)\right) \\
\tilde{K}_4^-(m_0|\lambda) &= \frac{e^{\mu\lambda}\kappa^-}{\sinh i\mu(\gamma^-+1)}\left(\frac{e^{i\mu\xi^-}}{2\kappa^-}\sinh i\mu(1+\gamma^-) - \frac{e^{-i\mu\xi^-}}{2\kappa^-}\sinh\mu(2\lambda+i\gamma^-+i)\right. \\
&\quad + \left. 2i\cosh i\mu(\beta^-+1)\sinh(2\mu\lambda)\right).
\end{aligned}
$$

$$\text{(B.2)}$$

By restricting our attention to the case where the left boundary is a trivial diagonal matrix (5.15), then the transformed diagonal entries become

$$\tilde{K}_1^+(m^0|\lambda) = -\frac{e^{-\mu\lambda}}{2\sinh(i\mu\gamma^+)}\sinh i\mu(1-\gamma^+), \quad \tilde{K}_4^+(m^0|\lambda) = \frac{e^{-\mu\lambda}}{2\sinh(i\mu\gamma^+)}\sinh i\mu(1+\gamma^+). \quad \text{(B.3)}$$

For the derivation of the spectrum we shall also need certain combinations of the diagonal transformed elements i.e.,

$$
\begin{aligned}
K_1^+(m|\lambda) &= \tilde{K}_1^+(m|\lambda) + \frac{\sinh\mu(i\gamma+m+2\lambda+i)\sinh(i\mu)}{\sinh i\mu(1+m+\gamma)\sinh\mu(2\lambda+i)}\tilde{K}_4^+(m|\lambda), \\
K_4^+(m|\lambda) &= \frac{\sinh i\mu(m+\gamma)\sinh(i\mu)}{\sinh i\mu(1+m+\gamma)\sinh\mu(2\lambda+i)}\tilde{K}_4^+(m|\lambda),
\end{aligned}
$$

$$\text{(B.4)}$$

$$
\begin{aligned}
K_1^-(m_0|\lambda) &= \tilde{K}_1^-(m_0|\lambda) \\
K_4^-(m_0|\lambda) &= \frac{\sinh i\mu(m_0+\gamma+1)\sinh\mu(2\lambda+i)}{\sinh i\mu(m_0+\gamma)\sinh(i\mu)}\tilde{K}_4^-(m_0|\lambda) \\
&\quad - \frac{\sinh\mu(im_0+i\gamma+2\lambda+i)}{\sinh i\mu(m_0+\gamma)}\tilde{K}_1^-(m_0|\lambda).
\end{aligned}
$$

$$\text{(B.5)}$$

It is instructive for our purposes here to consider the asymptotic behavior of $K_{1,4}^\pm$. We shall only deal with the case where K^+ is a trivial diagonal matrix (5.15). Considering the case where $\mu\lambda \to \infty$ we conclude that:

$$K_1^-(m_0|\mu\lambda \to \infty) = -i\kappa^- e^{3\mu\lambda - i\mu(\beta^-+\gamma^-)}, \quad K_4^-(m_0|\mu\lambda \to \infty) = -\frac{i\kappa^-}{2\sinh(i\mu)}e^{5\mu\lambda + i\mu(\beta^-+\gamma^-+2)}$$

$$K_1^+(m^0|\mu\lambda \to \infty) = e^{-\mu\lambda + i\mu}, \quad K_4^+(m^0|\mu\lambda \to \infty) = 2e^{-3\mu\lambda - i\mu}\sinh(i\mu). \quad \text{(B.6)}$$

C The boundary charge as a tridiagonal matrix: diagonalization

In this appendix we consider a particular example of the boundary operator $Q_1^{(N)}$ (5.5). We deal with its spin s representation (2.7) and for only one site. In this case the operator reduces to an $n \times n$ tridiagonal matrix (see also [35]) whose diagonalization will be the main objective in this Appendix

(A) Let us first present the boundary operator for one site Q_1 (5.5) in the spin s representation (2.7). Let us set for simplicity

$$A_k = q^{-\frac{1}{2}-\Theta_1} q^{\alpha_k} \tilde{C}_k, \qquad B_k = q^{\frac{1}{2}+\Theta_1} q^{\alpha_k} \tilde{C}_{k-1}, \qquad C_k = x_1 \, q^{2\alpha_k} \tag{C.1}$$

the boundary operator may be written as

$$Q_1 = \sum_{k=1}^{n-1} A_k \, e_{k \ k+1} + \sum_{k=2}^{n} B_k \, e_{k \ k-1} + \sum_{k=1}^{n} C_k \, e_{kk} \tag{C.2}$$

which is indeed a tridiagonal matrix. Let us now solve the corresponding eigenvalue problem. Let $\Psi = \sum_{l=1}^{n} w_l \, f_l$ be an eigenstate of Q_1 then

$$Q_1 \, \Psi = \epsilon \, \Psi \tag{C.3}$$

where ϵ is the corresponding eigenvalue. The latter equations can be written in a more explicit form as a tridiagonal (Jacobi) matrix,

$$\begin{pmatrix} C_1 & A_1 & & & & & \\ B_2 & C_2 & A_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & B_k & C_k & A_k & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & B_{n-1} & C_{n-1} & A_{n-1} \\ & & & & & B_n & C_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \\ \vdots \\ w_n \end{pmatrix} = \epsilon \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \\ \vdots \\ w_n \end{pmatrix} . \tag{C.4}$$

Note that in [35] the diagonalization of triadiagonal matrices and their relation to q hypergeometric series is discussed (also related to Leonard pairs, Askey-Wilson polynomials etc. see e.g. [59] and references therein).

One has now to solve the following recursion relations, in order to determine the factors w_l of the eigenstate,

$$A_1 \, w_2 = \hat{C}_1 \, w_1, \qquad A_k \, w_{k+1} + B_k \, w_{k-1} = \hat{C}_k \, w_k \quad k \in \{2, \ldots, n-1\}, \qquad B_n \, w_{n-1} = \hat{C}_n \, w_n \tag{C.5}$$

$\hat{C}_k = -C_k + \epsilon$. Finally solving the recursion relations we obtain compact expressions for the factors, i.e.

$$w_m = \prod_{j=1}^{m-1} \frac{\hat{C}_j}{A_j} + \sum_{k=1}^{l-1} (-)^k \sum_{j_k > j_{k-1} > \ldots > j_1 = 2, \ j_k - j_{k-1} > 1}^{m-1} \frac{\hat{C}_1 \ldots \hat{C}_{j_1-2} B_{j_1} \hat{C}_{j_1+1} \ldots \hat{C}_{j_k-2} B_{j_k} \hat{C}_{j_k+1} \ldots \hat{C}_{m-1}}{A_1 \ldots A_{j_1-2} A_{j_1} A_{j_1+1} \ldots A_{j_k-2} A_{j_k} A_{j_k+1} \ldots A_{m-1}},$$
$$m = 2l \quad \text{or} \quad m = 2l - 1. \tag{C.6}$$

(B) It is also worth discussing briefly the cyclic representation. In this case the structure of the boundary operator becomes more involved and its diagonalization a more intriguing task. In particular, the boundary operator is now of the form:

$$Q_1 = \sum_{k=1}^{p-1} A_k \ e_{k \ k+1} + A_p \ e_{p1} + \sum_{k=2}^{p} B_k \ e_{k \ k-1} + B_1 \ e_{1p} + \sum_{k}^{p} C_k \ e_{kk}. \tag{C.7}$$

Similarly we define:

$$A_k = q^{-\frac{1}{2}-\Theta_1-k} \ \frac{q^{-s-k} - q^{s+k}}{q - q^{-1}}, \qquad B_k = q^{\frac{1}{2}+\Theta_1-k} \ \frac{q^{-s+k} - q^{s-k}}{q - q^{-1}}, \qquad C_k = x_1 \ q^{-2k} \tag{C.8}$$

and the eigenvalue problem in a matrix form is then written as

$$\begin{pmatrix} C_1 & A_1 & & & & & B_1 \\ B_2 & C_2 & A_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & B_k & C_k & A_k & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & B_{n-1} & C_{n-1} & A_{n-1} \\ A_p & & & & & B_n & C_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \\ \vdots \\ w_n \end{pmatrix} = \epsilon \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \\ \vdots \\ w_n \end{pmatrix} \tag{C.9}$$

and finally to identify the spectrum and the corresponding eigenstates one has to solve the subsequent set of recursion relations:

$$A_k \ w_{k+1} + B_k \ w_{k-1} = \hat{C}_k \ w_k, \qquad k \in \{1, \ldots, p\}. \tag{C.10}$$

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